

The Birman-Wenzl-Murakami algebra, Hecke algebra and representations of $U_q(\mathfrak{osp}(1|2n))$

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Abstract

A representation of the Birman-Wenzl-Murakami algebra $BW_t(-q^{2n}, q)$ exists in the centraliser algebra $\text{End}_{U_q(\mathfrak{osp}(1|2n))}(V^{\otimes t})$, where V is the fundamental $(2n+1)$ -dimensional irreducible $U_q(\mathfrak{osp}(1|2n))$ -module. This representation is defined using permuted R -matrices acting on $V^{\otimes t}$. A complete set of projections onto and intertwiners between irreducible $U_q(\mathfrak{osp}(1|2n))$ -summands of $V^{\otimes t}$ exists via this representation, proving that $\text{End}_{U_q(\mathfrak{osp}(1|2n))}(V^{\otimes t})$ is generated by the set of permuting R -matrices acting on $V^{\otimes t}$.

We also show that a representation of the the Iwahori-Hecke algebra $H_t(-q)$ of type A_{t-1} exists in the centraliser algebra $\text{End}_{U_q(\mathfrak{osp}(1|2))}[(V_{1/2}^+)^{\otimes t}]$, where $V_{1/2}^+$ is a two-dimensional irreducible representation of $U_q(\mathfrak{osp}(1|2))$.

1 Introduction

The Birman-Wenzl-Murakami algebra, introduced independently by Murakami [19] and Birman and Wenzl [3] in connection with the study of the Kauffman knot polynomial, has been the subject of much research, including its relation with the centraliser algebras of repeated tensor products of the fundamental irreducible representations of the quantum algebras $U_q(\mathfrak{sp}(2n))$ and $U_q(\mathfrak{so}(2n+1))$ [5, 22]. Evidence has pointed towards a connection between the Birman-Wenzl-Murakami algebra and representations of $U_q(\mathfrak{osp}(1|2n))$ from a variety of sources, including the relation of the Brauer algebra, which the Birman-Wenzl-Murakami algebra can be seen as a deformation of, with tensor products of representations of $U(\mathfrak{osp}(1|2n))$ [2], and the isomorphism between tensorial irreducible representations of $U_{-q}(\mathfrak{so}(2n+1))$ and $U_q(\mathfrak{osp}(1|2n))$ [27].

Let $\mathfrak{g} = \mathfrak{osp}(1|2n)$ and V be the $(2n+1)$ -dimensional irreducible $U_q(\mathfrak{g})$ -module. In this paper we show that a subalgebra of the centraliser algebra $\text{End}_{U_q(\mathfrak{g})}(V^{\otimes t})$ yields a representation of the Birman-Wenzl-Murakami algebra $BW_t(-q^{2n}, q)$ and that this subalgebra, which is generated by a representation of the Braid group B_t on t strings, is in fact equal to $\text{End}_{U_q(\mathfrak{g})}(V^{\otimes t})$.

We also show a connection between the Hecke algebra $H_t(-q)$ of type A_{t-1} and repeated tensor powers of representations of $U_q(\mathfrak{osp}(1|2))$ that has not, to our knowledge, appeared in the literature.

It may be thought that the connection between $BW_t(-q^{2n}, q)$ and $\text{End}_{U_q(\mathfrak{g})}(V^{\otimes t})$ is a trivial consequence of the isomorphism between integrable tensorial representations of $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n+1))$ shown by Zhang [27]. However, this isomorphism is moreso indicative of such a connection and we define our representation of $BW_t(-q^{2n}, q)$ without reference to representations of any other quantum algebra.

The structure of this paper is as follows. In Section 2 we introduce the notation we use throughout this paper and necessary algebraic concepts, eg \mathbb{Z}_2 -graded vector spaces. In Section 3 we introduce the quantum superalgebra $U_q(\mathfrak{g})$. In Section 4 we discuss the finite-dimensional

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irreducible representations of $U_q(\mathfrak{osp}(1|2n))$, including the fundamental $(2n+1)$ -dimensional irreducible module V . We recall the isomorphism between finite-dimensional integrable tensorial representations of $U_q(\mathfrak{osp}(1|2n))$ and $U_{-q}(\mathfrak{so}(2n+1))$ and that $V^{\otimes t}$ is completely reducible for each t .

In Section 5 we introduce R -matrices for representations of $U_q(\mathfrak{g})$. In Section 6 we investigate the properties of two useful elements of a completion $\overline{U}_q^+(\mathfrak{g})$ of $U_q(\mathfrak{g})$. In Section 7 we determine the spectral decomposition of the element $\tilde{\mathcal{R}}_{V,V}$, which is a representation of a Braid group generator in $\text{End}_{U_q(\mathfrak{g})}(V \otimes V)$. In Section 8 we show that there is a representation of $BW_t(-q^{2n}, q)$ in the complex algebra \mathcal{C}_t generated by the $\tilde{\mathcal{R}}_{V,V}$ -matrices acting on the i^{th} and $(i+1)^{\text{st}}$ tensor powers of $V^{\otimes t}$ for $i = 1, \dots, t-1$. In Section 9 we recall the ideas of Bratteli diagrams and path algebras. In Section 10 we construct projections from $V^{\otimes t}$ onto all the irreducible $U_q(\mathfrak{g})$ -submodules of $V^{\otimes t}$ using elements of $\overline{U}_q^+(\mathfrak{g})$ from Section 6. In Section 11 we construct a complete set of matrix units in $\text{End}_{U_q(\mathfrak{g})}(V^{\otimes t})$ from matrix units in $BW_t(-q^{2n}, q)$ and prove that the algebra \mathcal{C}_t is in fact equal to $\text{End}_{U_q(\mathfrak{g})}(V^{\otimes t})$. In Section 12 we show that there exists a representation of $H_t(-q)$ in $\text{End}_{U_q(\mathfrak{osp}(1|2))} \left[(V_{1/2}^+)^{\otimes t} \right]$.

The work in this paper elucidates part of the author's Ph.D thesis [4] together with some further results. In writing this paper, the author became aware of the work of Lehrer and Zhang on 'strongly multiplicity free modules' of quantum algebras [17]. Their work may extend in a natural way to cover representations of quantum superalgebras, including part of the work in this paper, but we have not investigated this.

2 Preliminaires

Throughout, we fix \mathbb{Z} to be the integers, \mathbb{Z}_+ to be the non-negative integers and $\mathbb{N} = \mathbb{Z} \setminus \{0\}$. Let $q \in \mathbb{C}$ be not equal to 0 or 1, then for each $k \in \mathbb{Z}_+$ we define

$$[k]_q = \frac{1 - q^k}{1 - q}, \quad [k]_q! = [k]_q[k-1]_q \cdots [1]_q, \quad [0]_q! = 1.$$

A vector space W is said to be \mathbb{Z}_2 -graded if W can be expressed as a direct sum of vector subspaces: $W = W_0 \oplus W_1$. We say that an element $w \in W$ is homogeneous if $w \in W_0 \cup W_1$. Each homogeneous element $w \in W$ has a grading: w has an even grading (resp. an odd grading) if $w \in W_0$ (resp. $w \in W_1$). We denote the grading of the homogeneous element $w \in W$ by $[w] = i$, where $w \in W_i$.

Let V and W be \mathbb{Z}_2 -graded vector spaces. The graded permutation operator $P : V \otimes W \rightarrow W \otimes V$ is defined for homogeneous $v \in V$ and $w \in W$ by

$$P(v \otimes w) = (-1)^{[v][w]} w \otimes v,$$

and is extended to inhomogeneous elements of V and W by linearity.

Let $V = \bigoplus_{i \in \mathbb{Z}_2} V_i$ and $W = \bigoplus_{j \in \mathbb{Z}_2} W_j$ be \mathbb{Z}_2 -graded vector spaces. A \mathbb{C} -linear map $\psi : V \rightarrow W$ is said to be homogeneous of degree $k \in \mathbb{Z}_2$ if $\psi(V_i) \subseteq W_{i+k}$ for each $i \in \mathbb{Z}_2$. The vector space $\text{Hom}_{\mathbb{C}}(V, W)$ of \mathbb{C} -linear maps from V to W admits a \mathbb{Z}_2 -gradation: $\text{Hom}_{\mathbb{C}}(V, W) = \bigoplus_{i \in \mathbb{Z}_2} \text{Hom}_{\mathbb{C}}(V, W)_i$ where $\text{Hom}_{\mathbb{C}}(V, W)_i = \{\psi \in \text{Hom}_{\mathbb{C}}(V, W) \mid \psi \text{ is homogeneous of degree } i\}$. We fix $\text{End}_{\mathbb{C}}(V) = \text{Hom}_{\mathbb{C}}(V, V)$.

The dual space to V is $V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ where we regard \mathbb{C} as a \mathbb{Z}_2 -graded vector space with zero odd subspace. V^* is \mathbb{Z}_2 -graded.

One can define \mathbb{Z}_2 -graded algebras, \mathbb{Z}_2 -graded co-algebras, and \mathbb{Z}_2 -graded Hopf algebras in a standard way (see [9] for further information).

Let A be a \mathbb{Z}_2 -graded algebra. A homomorphism of \mathbb{Z}_2 -graded A -modules is by definition a homomorphism of the A -modules as well as of the underlying \mathbb{Z}_2 -graded vector spaces. Each such homomorphism is A -linear and homogeneous of degree 0. Let $V = V_0 \oplus V_1$ be a \mathbb{Z}_2 -graded

A -module. If $V' = V'_0 \oplus V'_1$ with $V'_0 = V_1$ and $V'_1 = V_0$ then we regard V and V' as not being isomorphic.

Throughout this paper we fix $\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$

3 The quantum superalgebra $U_q(osp(1|2n))$

In this section we introduce the quantum superalgebra $U_q(osp(1|2n))$ [12, 25, 26]. Let us begin by describing the root system of $osp(1|2n)$. Let H^* be a vector space over \mathbb{C} with a basis $\{\epsilon_i \mid 1 \leq i \leq n\}$ and let

$$(\cdot, \cdot) : H^* \times H^* \rightarrow \mathbb{C}, \quad (1)$$

be a non-degenerate bilinear form defined by $(\epsilon_i, \epsilon_j) = \delta_{ij}$. The set of simple roots of $osp(1|2n)$ is $\{\alpha_i \mid 1 \leq i \leq n\}$ where

$$\alpha_i = \begin{cases} \epsilon_i - \epsilon_{i+1}, & i = 1, \dots, n-1, \\ \epsilon_n, & i = n, \end{cases}$$

which forms another basis of H^* .

The set of the positive roots of $osp(1|2n)$ is

$$\Phi^+ = \{\epsilon_i \pm \epsilon_j, \epsilon_k, 2\epsilon_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\},$$

and we further define the subsets of positive even roots (resp. positive odd roots) Φ_0^+ (resp. Φ_1^+) of Φ^+ and also a subset $\overline{\Phi}_0^+ \subset \Phi_0^+$ by

$$\begin{aligned} \Phi_0^+ &= \{\epsilon_i \pm \epsilon_j, 2\epsilon_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\}, & \Phi_1^+ &= \{\epsilon_k \mid 1 \leq k \leq n\}, \\ \overline{\Phi}_0^+ &= \{\alpha \in \Phi_0^+ \mid \alpha/2 \notin \Phi_1^+\}. \end{aligned}$$

The set of negative roots of $osp(1|2n)$ is $\Phi^- = -\Phi^+$, and $\Phi = \Phi^+ \cup \Phi^-$ is the set of all roots of $osp(1|2n)$.

We denote by $2\rho \in H^*$ the graded sum of the positive roots of $osp(1|2n)$:

$$2\rho = \sum_{\alpha \in \Phi_0^+} \alpha - \sum_{\beta \in \Phi_1^+} \beta.$$

Explicitly, $2\rho = \sum_{i=1}^n (2n-2i+1)\epsilon_i$. The element 2ρ satisfies $(2\rho, \alpha_i) = (\alpha_i, \alpha_i)$ for each $1 \leq i \leq n$.

The Cartan matrix of $osp(1|2n)$ is $A = (a_{ij})_{i,j=1}^n$ where $a_{ij} = 2(\alpha_i, \alpha_j) / (\alpha_i, \alpha_i)$:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -2 & 2 \end{pmatrix}.$$

The Lie superalgebra $\mathfrak{g} = osp(1|2n)$ over \mathbb{C} can be defined in terms of a Serre presentation with generators $\{E_i, F_i, H_i \mid 1 \leq i \leq n\}$ subject to the relations

$$\begin{aligned} [E_i, F_j] &= \delta_{ij} H_i, & [H_i, H_j] &= 0, & \forall i, j, \\ [H_i, E_j] &= (\alpha_i, \alpha_j) E_j, & [H_i, F_j] &= -(\alpha_i, \alpha_j) F_j, & \forall i, j, \\ (\text{ad } E_i)^{1-a_{ij}} E_j &= 0, & (\text{ad } F_i)^{1-a_{ij}} F_j &= 0, & \forall i \neq j, \end{aligned} \quad (2)$$

where $[\cdot, \cdot]$ denotes the \mathbb{Z}_2 -graded Lie bracket and the adjoint operation is $(\text{ad } a)b = [a, b]$. The \mathbb{Z}_2 -grading of the generators is

$$[E_i] = [F_i] = [H_j] = 0, \quad [E_n] = [F_n] = 1, \quad 1 \leq i \leq n-1, \quad 1 \leq j \leq n.$$

The universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} is a unital associative \mathbb{Z}_2 -graded algebra, which may be considered as being generated by $\{E_i, F_i, H_i \mid 1 \leq i \leq n\}$ subject to relations that are formally the same as (2) but with the bracket $[\cdot, \cdot]$ interpreted as a \mathbb{Z}_2 -graded commutator $[\cdot, \cdot] : U(\mathfrak{g}) \times U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ defined by

$$[X, Y] = XY - (-1)^{[X][Y]} YX. \quad (3)$$

If each of two elements $X, Y \in U(\mathfrak{g})$ has a grading, then the grading of $XY \in U(\mathfrak{g})$ is defined by

$$[XY] = ([X] + [Y]) \bmod 2.$$

The graded commutator $[X, Y]$ of any two homogeneous elements of $U(\mathfrak{g})$ is defined by (3) and is extended to inhomogeneous elements of $U(\mathfrak{g})$ by linearity. Note that $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ has a natural \mathbb{Z}_2 -graded associative algebra structure, with the grading defined for homogeneous $X, Y \in U(\mathfrak{g})$ by

$$[X \otimes Y] = ([X] + [Y]) \bmod 2.$$

The quantum superalgebra $U_q(\mathfrak{g})$ is some “ q -deformation” of $U(\mathfrak{g})$. We describe its Jimbo version here.

Definition 3.1. *The quantum superalgebra $U_q(\mathfrak{g})$ over \mathbb{C} is an associative \mathbb{Z}_2 -graded unital algebra generated by $\{e_i, f_i, K_i, K_i^{-1} \mid 1 \leq i \leq n\}$ subject to the relations*

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\ K_i e_j K_i^{-1} &= q^{(\alpha_i, \alpha_j)} e_j, \quad K_i f_j K_i^{-1} = q^{-(\alpha_i, \alpha_j)} f_j, \\ [K_i^{\pm 1}, K_j^{\pm 1}] &= [K_i^{\pm 1}, K_j^{\mp 1}] = 0, \quad K_i^{\pm 1} K_i^{\mp 1} = 1, \\ (\text{ad}_q e_i)^{1-a_{ij}} e_j &= 0, \quad (\text{ad}_q f_i)^{1-a_{ij}} f_j = 0, \quad \forall i, j, \end{aligned} \quad (4)$$

where $0 \neq q \in \mathbb{C}$ and $q^2 \neq 1$, and the adjoint actions are defined by

$$\begin{aligned} (\text{ad}_q e_i)X &= e_i X - (-1)^{[e_i][X]} K_i X K_i^{-1} e_i, \\ (\text{ad}_q f_i)X &= f_i X - (-1)^{[f_i][X]} K_i^{-1} X K_i f_i, \end{aligned}$$

for all homogeneous $X \in U_q(\mathfrak{g})$. The grading of each generator is even except for e_n and f_n , which are odd. In (4), the bracket $[\cdot, \cdot]$ is as defined in (3).

For each $\beta = \sum_{i=1}^n m_i \alpha_i$ where $m_i \in \mathbb{Z}$, we define $K_\beta = \prod_{i=1}^n (K_i)^{m_i}$.

As is well known, there exists a \mathbb{Z}_2 -graded Hopf algebra structure on $U_q(\mathfrak{g})$ with the co-multiplication Δ , the co-unit ϵ , and the antipode S defined on each generator by

$$\begin{aligned} \Delta(e_i) &= e_i \otimes K_i + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i^{-1} \otimes f_i, \quad \Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \\ \epsilon(e_i) &= \epsilon(f_i) = 0, \quad \epsilon(K_i^{\pm 1}) = \epsilon(1) = 1, \\ S(e_i) &= -e_i K_i^{-1}, \quad S(f_i) = -K_i f_i, \quad S(K_i^{\pm 1}) = K_i^{\mp 1}. \end{aligned}$$

There are a number of quite useful subalgebras of $U_q(\mathfrak{g})$. We define $U_q(\mathfrak{b}_+)$ (resp. $U_q(\mathfrak{b}_-)$) to be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{e_i, K_i^{\pm 1} \mid 1 \leq i \leq n\}$ (resp. $\{f_i, K_i^{\pm 1} \mid 1 \leq i \leq n\}$).

We shall sometimes refer to the quantum superalgebra in the sense of Drinfel'd which we denote by $U_h(\mathfrak{g})$. This is a \mathbb{Z}_2 -graded Hopf algebra over the ring $\mathbb{C}[[h]]$ in an indeterminate h ,

completed with respect to the h -adic topology [12]. Fix $q = e^h$. The algebra $U_h(\mathfrak{g})$ is generated by $\{E_i, F_i, H_i \mid 1 \leq i \leq n\}$ with relations

$$\begin{aligned} [E_i, F_j] &= \delta_{ij} \frac{e^{hH_i} - e^{-hH_i}}{e^h - e^{-h}}, & [H_i, H_j] &= 0, \\ [H_i, E_j] &= (\alpha_i, \alpha_j) E_j, & [H_i, F_j] &= -(\alpha_i, \alpha_j) F_j, \\ (\text{ad}_q E_i)^{1-a_{ij}} E_j &= 0, & (\text{ad}_q F_i)^{1-a_{ij}} F_j &= 0, \quad \forall i, j, \end{aligned}$$

where the adjoint functions are defined by

$$\begin{aligned} (\text{ad}_q E_i)X &= E_i X - (-1)^{[E_i][X]} e^{hH_i} X e^{-hH_i} E_i, \\ (\text{ad}_q F_i)X &= F_i X - (-1)^{[F_i][X]} e^{-hH_i} X e^{hH_i} F_i. \end{aligned}$$

A \mathbb{Z}_2 -graded quasitriangular Hopf algebra A is a \mathbb{Z}_2 -graded Hopf algebra that admits a universal R -matrix, which is an invertible even element $R \in A \otimes A$ satisfying the relations

$$R\Delta(x) = \Delta'(x)R, \quad \forall x \in A \quad (5)$$

$$(\Delta \otimes \text{id})R = R_{13}R_{23} \quad (6)$$

$$(\text{id} \otimes \Delta)R = R_{13}R_{12} \quad (7)$$

where upon writing $R = \sum_s \alpha_s \otimes \beta_s$, we write $R_{12} = R \otimes \text{id}$, $R_{13} = \sum_s \alpha_s \otimes \text{id} \otimes \beta_s$ and $R_{23} = \text{id} \otimes R$, and also write $\Delta'(x) = P(\Delta(x))$ where P is the graded permutation operator. It is important to note that $U_h(\mathfrak{g})$ admits a universal R -matrix [12] and is a \mathbb{Z}_2 -graded ribbon Hopf algebra [29].

Given a finite-dimensional $U_q(\mathfrak{g})$ -module W , we define the quantum supertrace of $f \in \text{End}_{\mathbb{C}}(W)$ to be

$$\text{str}_q(f) = \text{str}(\pi_W(K_{2\rho}) \cdot f).$$

4 Finite dimensional irreducible $U_q(\text{osp}(1|2n))$ -modules

Given a \mathbb{Z}_2 -graded algebra A , we will denote by V_λ an A -module labelled by $\lambda \in I$ for some index set I , and we denote by π_λ the representation of A afforded by V_λ .

In this section we assume that q is non-zero and not a root of unity; in this case the representation theory of $U_q(\mathfrak{g})$ is completely understood [1, 27, 30].

We say that an element $\lambda \in H^*$ is *integral* if

$$l_i = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}, \quad \forall i < n, \quad l_n = \frac{(\lambda, \alpha_n)}{(\alpha_n, \alpha_n)} \in \mathbb{Z},$$

where $(\cdot, \cdot) : H^* \times H^* \rightarrow \mathbb{C}$ is the bilinear form from (1) and we let \mathcal{P} be the set of all integral elements of H^* . Furthermore, we say that an element $\lambda \in \mathcal{P}$ is *integral dominant* if $l_i \in \mathbb{Z}_+$ for all i and denote by \mathcal{P}^+ the set of all integral dominant elements of H^* .

Zou investigated the representation theory of the quantum superalgebra $U_q(\mathfrak{g})$ over the quotient field $\mathbb{C}(v)$ for an indeterminate v [30], which is related to q via $q = v^2$. Zou's results can be adapted to our setting where we take q to be generic. Let \sqrt{q} be any square root of the complex number q . Call a $U_q(\mathfrak{g})$ -module V *integrable* if V is a direct sum of its weight spaces and if e_i and f_i act as locally nilpotent endomorphisms of V for each $i = 1, \dots, n$.

Let $\overline{V}(\omega)$ be a highest weight $U_q(\mathfrak{g})$ -module with highest weight vector v satisfying $K_i v = \omega_i v$, $\omega_i \in \mathbb{C}$, for each $i = 1, \dots, n$, then $\overline{V}(\omega)$ has a unique maximal proper $U_q(\mathfrak{g})$ -submodule $\overline{M}(\omega)$, and the quotient

$$V(\omega) = \overline{V}(\omega) / \overline{M}(\omega)$$

is an irreducible $U_q(\mathfrak{g})$ -module with highest weight $\omega = (\omega_1, \omega_2, \dots, \omega_n)$. Theorem 3.1 of [30] can be stated in our setting as follows.

Theorem 4.1. *The irreducible highest weight $U_q(\mathfrak{g})$ -module $V(\omega)$, with $\omega = (\omega_1, \omega_2, \dots, \omega_n)$, is integrable if and only if*

$$\omega_i = \zeta_i q^{m_i}, \quad 1 \leq i \leq n-1,$$

where $m_i \in \mathbb{Z}_+$, $\zeta_i^2 = 1$, and

$$\omega_n = \begin{cases} \pm q^m, & \text{if } m \in \mathbb{Z}_+, \\ \pm \sqrt{-1} q^m, & \text{if } m \in \mathbb{Z}_+ + \frac{1}{2}. \end{cases}$$

Note that every finite dimensional integrable $U_q(\mathfrak{g})$ -module is semisimple [30, Sec. 5].

If $\omega = (q^{m_1}, q^{m_2}, \dots, q^{m_n})$ with $m_i \in \mathbb{Z}_+$ for each i , there exists an irreducible $U(\mathfrak{g})$ -module $V(\omega)_{\mathfrak{g}}$ with highest weight $\lambda \in \mathcal{P}^+$ satisfying $(\lambda, \alpha_i) = m_i$ for each i , and $V(\omega)$ and $V(\omega)_{\mathfrak{g}}$ have the same weight space decompositions [27]. In this paper, we are more interested in these irreducible $U_q(\mathfrak{g})$ -modules, and for each $\lambda \in \mathcal{P}^+$ we let V_λ denote $V(\omega)$ and call λ the highest weight of V_λ .

For convenience we fix the grading of the highest weight vector of the finite dimensional irreducible $U_q(\mathfrak{g})$ -module V_λ with integral dominant highest weight $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \mathcal{P}^+$ to be even (resp. odd) if $\sum_{i=1}^n \lambda_i$ is even (resp. odd).

The following two lemmas restate some important results from [27].

Lemma 4.1. *Let $\mu \in \mathcal{P}^+$ and let V_μ be a finite dimensional irreducible $U_q(\mathfrak{osp}(1|2n))$ -module with highest weight μ . Then V_μ is also an irreducible $U_{-q}(\mathfrak{so}(2n+1))$ -module with highest weight μ . The weight space decompositions of V_μ as a $U_q(\mathfrak{osp}(1|2n))$ -module and as a $U_{-q}(\mathfrak{so}(2n+1))$ -module are the same. Furthermore, the quantum dimension of V_μ as a $U_{-q}(\mathfrak{so}(2n+1))$ -module equals the quantum superdimension of V_μ as a $U_q(\mathfrak{osp}(1|2n))$ -module.*

Let $\mu, \nu \in \mathcal{P}^+$ and let V_μ, V_ν be irreducible $U_{-q}(\mathfrak{so}(2n+1))$ -modules with highest weights μ and ν , respectively, then it is well known that $V_\mu \otimes V_\nu$ is completely reducible into a direct sum of irreducible $U_{-q}(\mathfrak{so}(2n+1))$ -submodules:

$$V_\mu \otimes V_\nu \cong \bigoplus_{\lambda \in \mathcal{P}^+} \left(\mathbb{C}^{m(\lambda, \mu, \nu)} \otimes V_\lambda \right), \quad (8)$$

where $m(\lambda, \mu, \nu) \geq 0$ is the number of copies of irreducible $U_{-q}(\mathfrak{so}(2n+1))$ -submodules of $V_\mu \otimes V_\nu$ in the decomposition isomorphic to V_λ . Then:

Lemma 4.2. *Let $\mu, \nu \in \mathcal{P}^+$ and let V_μ, V_ν be irreducible $U_q(\mathfrak{osp}(1|2n))$ -modules with highest weights μ and ν , respectively, then $V_\mu \otimes V_\nu$ is completely reducible into a direct sum of irreducible $U_q(\mathfrak{osp}(1|2n))$ -submodules:*

$$V_\mu \otimes V_\nu \cong \bigoplus_{\lambda \in \mathcal{P}^+} \left(\mathbb{C}^{m(\lambda, \mu, \nu)} \otimes V_\lambda \right),$$

where $m(\lambda, \mu, \nu) \geq 0$ is the number of copies of irreducible $U_q(\mathfrak{osp}(1|2n))$ -submodules of $V_\mu \otimes V_\nu$ in the decomposition isomorphic to V_λ , where the constants $m(\lambda, \mu, \nu)$ are the same as in (8).

Let W be a $U_q(\mathfrak{g})$ -module with homogeneous basis $\{w_i\}_{i \in I}$. Let $\{w_i^*\}$ be a basis of W^* where $[w_i^*] = [w_i]$ and the dual space pairing is $\langle w_i^*, w_j \rangle = \delta_{ij}$. Then W^* is the dual $U_q(\mathfrak{g})$ -module to W with the action of $U_q(\mathfrak{g})$ given by $\langle aw_i^*, w_j \rangle = (-1)^{[a][w_i^*]} \langle w_i^*, S(a)w_j \rangle$, $\forall a \in U_q(\mathfrak{g})$.

We now introduce the fundamental (or vector) $U_q(\mathfrak{g})$ -module V in the next lemma adapted from [16], which we state without proof.

Lemma 4.3. *There exists a $(2n+1)$ -dimensional irreducible $U_q(\mathfrak{osp}(1|2n))$ -module $V = V_{\epsilon_1}$ with highest weight ϵ_1 . This module admits a basis $\{v_i \mid -n \leq i \leq n\}$ with v_1 being the highest weight vector. The actions of the generators of $U_q(\mathfrak{osp}(1|2n))$ on the basis elements are*

$$\begin{aligned} f_i v_i &= v_{i+1}, & f_n v_n &= v_0, & f_n v_0 &= v_{-n}, & f_i v_{-i-1} &= v_{-i}, \\ e_i v_{i+1} &= v_i, & e_n v_0 &= v_n, & e_n v_{-n} &= -v_0, & e_i v_{-i} &= v_{-i-1}, \\ K_j^{\pm 1} v_k &= q^{\pm(\alpha_j, \epsilon_k)} v_k, \end{aligned}$$

where $1 \leq i < n$, $1 \leq j \leq n$, $-n \leq k \leq n$, and we fix $\epsilon_0 = 0$, and $\epsilon_{-i} = -\epsilon_i$. All remaining actions are zero.

Note that the highest weight vector v_1 of V has an odd grading.

Proposition 4.1. *There exists a $U_q(\mathfrak{g})$ -invariant, non-degenerate bilinear form $\langle\langle \cdot, \cdot \rangle\rangle : V \times V \rightarrow \mathbb{C}$. Thus the dual $U_q(\mathfrak{g})$ -module of V is isomorphic to V .*

Proof. Let $\{v_i \mid -n \leq i \leq n\}$ be the basis of V given in Lemma 4.3. Now define a non-degenerate \mathbb{C} -bilinear form $\langle\langle \cdot, \cdot \rangle\rangle : V \times V \rightarrow \mathbb{C}$ by

$$\begin{aligned} \langle\langle v_1, v_{-1} \rangle\rangle &= 1, \\ \langle\langle v_i, v_{-i} \rangle\rangle &= -q^{-1} \langle\langle v_{i-1}, v_{-(i-1)} \rangle\rangle, \quad 2 \leq i \leq n, \\ \langle\langle v_0, v_0 \rangle\rangle &= q^{-1} \langle\langle v_n, v_{-n} \rangle\rangle, \\ \langle\langle v_{-n}, v_n \rangle\rangle &= -\langle\langle v_0, v_0 \rangle\rangle, \\ \langle\langle v_{-j}, v_j \rangle\rangle &= -q^{-1} \langle\langle v_{-(j+1)}, v_{j+1} \rangle\rangle, \quad 1 \leq j \leq n-1, \\ \langle\langle v_k, v_l \rangle\rangle &= 0, \quad \text{if } k+l \neq 0. \end{aligned}$$

A direct calculation shows that

$$\langle\langle xv_i, v_j \rangle\rangle = (-1)^{[x][v_i]} \langle\langle v_i, S(x)v_j \rangle\rangle, \quad \forall x \in U_q(\mathfrak{g}), \quad v_i, v_j \in V,$$

thus proving the $U_q(\mathfrak{g})$ -invariance of the bilinear form. This form identifies V with its dual module. \square

Let us discuss in more detail the dual module of V . Recall the definition of the dual $U_q(\mathfrak{g})$ -module V^* to V . Let $\{v_i^* \mid -n \leq i \leq n\}$ be a basis of V^* such that $\langle v_i^*, v_j \rangle = \delta_{ij}$ and $[v_i^*] = [v_i]$ where $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{C}$ is the dual space pairing. Now define a homogeneous bijection $T \in \text{Hom}_{\mathbb{C}}(V, V^*)$ of degree 0 for all $1 \leq i \leq n$ by

$$T : v_i \mapsto (-1)^{i-1} q^{-(i-1)} v_{-i}^*, \quad v_0 \mapsto (-1)^{n-1} q^{-n} v_0^*, \quad v_{-i} \mapsto (-1)^i q^{-(2n-i)} v_i^*. \quad (9)$$

A direct calculation shows that this map is an element of $\text{Hom}_{U_q(\mathfrak{g})}(V, V^*)$ and that it satisfies $\langle T(v_i), v_j \rangle = \langle\langle v_i, v_j \rangle\rangle$ for all $v_i, v_j \in V$.

5 R -matrices for representations of $U_q(\text{osp}(1|2n))$

Drinfel'd's quantum superalgebra $U_h(\mathfrak{g})$ admits a universal R -matrix [12, 25]. We will show that there does not exist an element of $U_q(\mathfrak{g})$ that corresponds to the universal R -matrix of $U_h(\mathfrak{g})$ in any obvious way. However, there exists a completion $\overline{U}_q^+(\mathfrak{g})$ of $U_q(\mathfrak{g})$ such that one of the multiplicative factors of the universal R -matrix of $U_h(\mathfrak{g})$ maps to an element \tilde{R} of $\overline{U}_q^+(\mathfrak{g})$. Although \tilde{R} is not an element of $U_q(\mathfrak{g})$, only a finite number of its terms act as non-zero endomorphisms on each tensor product of finite dimensional irreducible $U_q(\mathfrak{g})$ -modules, and so the action of \tilde{R} on such tensor products is well-defined. This allows us to define R -matrices for representations of $U_q(\mathfrak{g})$ later in this paper.

5.1 The universal R -matrix of $U_h(\mathfrak{g})$

Khoroshkin and Tolstoy [12] wrote down a universal R -matrix of $U_h(\mathfrak{g})$ using infinite sums of root vectors in $U_h(\mathfrak{g})$. Yamane also wrote down a universal R -matrix of $U_h(\mathfrak{g})$ [25] but we use Khoroshkin and Tolstoy's work for ease. The root vectors in [12] are defined in a different way to the way in which root vectors in universal R -matrices of quantum algebras are defined. Khoroshkin and Tolstoy's procedure is general for quantum superalgebras and we write it down here for $U_h(\mathfrak{g})$.

For the universal R -matrix of $U_h(\mathfrak{g})$, we only define root vectors for the elements of the *reduced root system* ϕ of \mathfrak{g} . The reduced root system ϕ is the set of all positive roots of \mathfrak{g} except those roots α for which $\alpha/2$ is also a positive root: the reduced root system of $\mathfrak{g} = \text{osp}(1|2n)$ is $\phi = \overline{\Phi}_0^+ \cup \Phi_1^+ = \{\epsilon_i \pm \epsilon_j, \epsilon_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\}$.

We introduce a total ordering of ϕ called a *normal ordering* which we denote by $\mathcal{N}(\phi)$, and then recursively define the root vectors of $U_h(\mathfrak{g})$ using $\mathcal{N}(\phi)$ and a map involving the q -bracket we introduce below. The way in which the universal R -matrix of $U_h(\mathfrak{g})$ is formally written down explicitly depends on $\mathcal{N}(\phi)$. A difference between the root vectors in quantum algebras and the root vectors in $U_q(\mathfrak{g})$ is that the latter are defined by a map that is not necessarily an algebra automorphism.

A normal ordering of the reduced root system of \mathfrak{g} is defined as follows [12, Def. 3.1].

Definition 5.1. A normal ordering $\mathcal{N}(\phi)$ of $\phi = \overline{\Phi}_0^+ \cup \Phi_1^+$ is a total order \prec of the elements of ϕ such that if $\alpha \prec \beta$ and $\alpha + \beta \in \phi$, then $\alpha \prec \alpha + \beta \prec \beta$.

In general, there is more than one normal ordering of ϕ [12]. For example, the reduced root system of $osp(1|4)$ is $\phi = \{\epsilon_1, \epsilon_2, \epsilon_1 \pm \epsilon_2\}$ and there are two different normal orderings of ϕ :

$$\alpha_1 \prec \alpha_1 + \alpha_2 \prec \alpha_1 + 2\alpha_2 \prec \alpha_2,$$

$$\alpha_2 \prec \alpha_1 + 2\alpha_2 \prec \alpha_1 + \alpha_2 \prec \alpha_1,$$

writing the elements of ϕ as sums of the simple roots.

The universal R -matrix of $U_h(\mathfrak{g})$ [12], adapted slightly to take account of the different co-multiplication used in this paper, is as follows. Let us write $q = e^h \in \mathbb{C}[[h]]$, and $\exp_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}$.

Construct the root vectors as follows. Firstly fix $E_{\alpha_i} = E_i$, $F_{\alpha_i} = F_i$ and $H_{\alpha_i} = H_i$ for each simple root α_i . Now recursively construct the non-simple root vectors. Let $\alpha, \beta, \gamma \in \phi$ be roots such that $\gamma = \alpha + \beta$ and $\alpha \prec \beta$, and furthermore let no other roots $\alpha', \beta' \in \phi$ exist which satisfy $\alpha' + \beta' = \gamma$, $\alpha \prec \alpha' \prec \beta$ and $\alpha \prec \beta' \prec \beta$. Then, if all of the root vectors $E_\alpha, E_\beta, F_\alpha, F_\beta \in U_h(\mathfrak{g})$ have already been defined, fix

$$E_\gamma = [E_\alpha, E_\beta]_q, \quad F_\gamma = [F_\beta, F_\alpha]_{q^{-1}},$$

where the q -bracket $[\cdot, \cdot]_q$ is defined by

$$[X_\alpha, X_\beta]_q = X_\alpha X_\beta - (-1)^{[X_\alpha][X_\beta]} q^{(\alpha, \beta)} X_\beta X_\alpha,$$

where we replace X with E or F as appropriate.

Now for each $\gamma \in \phi$, fix

$$\tilde{R}_\gamma = \exp_{q_\gamma}((-1)^{[E_\gamma]}(a_\gamma)^{-1}(q - q^{-1})E_\gamma \otimes F_\gamma) \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g}), \quad (10)$$

where $q_\gamma = (-1)^{[E_\gamma]} q^{-(\gamma, \gamma)}$ and $a_\gamma \in \mathbb{C}[[h]]$ is defined by

$$E_\gamma F_\gamma - (-1)^{[E_\gamma]} F_\gamma E_\gamma = \frac{a_\gamma (q^{H_\gamma} - q^{-H_\gamma})}{q - q^{-1}}.$$

It is important to observe that a_γ is a rational function of q . Now we can write down the universal R -matrix of $U_h(\mathfrak{g})$ [12, Thm. 8.1].

Theorem 5.1. Define $\mathbf{H}_i = \sum_{j=i}^n H_j \in U_h(\mathfrak{g})$ for each $i = 1, \dots, n$. The universal R -matrix of $U_h(\mathfrak{g})$ is

$$R = \exp \left(h \sum_{i=1}^n \mathbf{H}_i \otimes \mathbf{H}_i \right) \prod_{\gamma \in \phi} \tilde{R}_\gamma, \quad (11)$$

where the product is ordered with respect to the same normal ordering $\mathcal{N}(\phi)$ that was used to define the root vectors in $U_h(\mathfrak{g})$ so that $\prod_{\gamma \in \phi} \tilde{R}_\gamma = \tilde{R}_{\gamma_1} \tilde{R}_{\gamma_2} \cdots \tilde{R}_{\gamma_k}$ if the normal ordering $\mathcal{N}(\phi)$ is $\gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_k$.

5.2 R -matrices for representations of $U_q(\mathfrak{g})$

It is unknown whether Jimbo's quantum algebras admit universal R -matrices. However, it is well-known that there are R -matrices for representations of these quantum algebras. Let π_λ and π_μ be finite dimensional irreducible representations of the quantum algebra A , then there exists an invertible element $\mathcal{R}_{\lambda\mu} \in \text{End}_{\mathbb{C}}(V_\lambda \otimes V_\mu)$ satisfying

$$\mathcal{R}_{\lambda\mu} \cdot (\pi_\lambda \otimes \pi_\mu)(\Delta(x)) = (\pi_\lambda \otimes \pi_\mu)(\Delta'(x)) \cdot \mathcal{R}_{\lambda\mu} \quad \forall x \in A. \quad (12)$$

We will show that for each tensor product $V_\lambda \otimes V_\mu$ of finite dimensional irreducible $U_q(\mathfrak{g})$ -modules, there exists a map $\mathcal{R}_{\lambda\mu} \in \text{End}_{\mathbb{C}}(V_\lambda \otimes V_\mu)$ satisfying (12) for all $x \in U_q(\mathfrak{g})$. We will do this following the method shown in [5, 13] for representations of quantum algebras.

We firstly define a completion $\overline{U}_q^+(\mathfrak{g})$ of $U_q(\mathfrak{g})$. Let $U_q(\mathfrak{n}_+)$ (resp. $U_q(\mathfrak{n}_-)$) be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{e_i \mid 1 \leq i \leq n\}$ (resp. $\{f_i \mid 1 \leq i \leq n\}$). We say that a non-zero element $x \in U_q(\mathfrak{g})$ has degree $\lambda = \sum_{i=1}^n m_i \alpha_i$, $m_i \in \mathbb{Z}$, if $K_i x K_i^{-1} = q^{(\lambda, \alpha_i)} x$ for all $i = 1, 2, \dots, n$. We define $\overline{U}_q^\pm(\mathfrak{g})$ by

$$\overline{U}_q^\pm(\mathfrak{g}) = \prod_{\beta \in Q_+} U_q(\mathfrak{b}_\pm) U_q^{\mp\beta}(\mathfrak{n}_\mp),$$

where $U_q^{\mp\beta}(\mathfrak{n}_\mp)$ is defined by

$$U_q^{\pm\beta}(\mathfrak{n}_\pm) = \{x \in U_q(\mathfrak{n}_\pm) \mid K_i x K_i^{-1} = q^{\pm(\alpha_i, \beta)} x\}, \quad i = 1, 2, \dots, n,$$

and Q_+ is defined by $Q_+ = \{\sum_{i=1}^n n_i \alpha_i \mid n_i \in \mathbb{Z}_+\}$.

The elements of $\overline{U}_q^\pm(\mathfrak{g})$ are sequences $x = (x_\beta)_{\beta \in Q_+}$ where $x_\beta \in U_q(\mathfrak{b}_\pm) U_q^{\mp\beta}(\mathfrak{n}_\mp)$. Let us write this sequence formally as an infinite sum $x = \sum_{\beta} x_\beta$. Then $U_q(\mathfrak{g})$ can be expressed as

$$U_q(\mathfrak{g}) = \bigoplus_{\beta \in Q_+} U_q(\mathfrak{b}_+) U_q^{-\beta}(\mathfrak{n}_-),$$

thus $U_q(\mathfrak{g})$ can be considered as the subspace of $\overline{U}_q^+(\mathfrak{g})$ formed by the sums $x = \sum_{\beta} x_\beta$ for which all but finitely many terms vanish.

The multiplication in $U_q(\mathfrak{g})$ extends to multiplications in $\overline{U}_q^\pm(\mathfrak{g})$, each of which is an algebra, as is $\overline{U}_q^\pm(\mathfrak{g}) \otimes \dots \otimes \overline{U}_q^\pm(\mathfrak{g})$ (m factors), and these algebras, respectively, contain $U_q(\mathfrak{g})$ and $U_q(\mathfrak{g}) \otimes \dots \otimes U_q(\mathfrak{g})$ (m factors) as subalgebras.

We now construct an element in $\overline{U}_q^+(\mathfrak{g})$ corresponding to $\prod_{\gamma \in \phi} \tilde{R}_\gamma$ in (11). Given a normal ordering $\mathcal{N}(\phi)$ for a reduced root system ϕ , we construct root vectors $E_\gamma, F_\gamma \in U_q(\mathfrak{g})$ following the same procedure as in $U_h(\mathfrak{g})$ by setting $E_{\alpha_i} = e_i$ and $F_{\alpha_i} = f_i$ and thinking of q a complex number. Each \tilde{R}_γ is then well-defined as an element of $\overline{U}_q^+(\mathfrak{g}) \otimes \overline{U}_q^+(\mathfrak{g})$, and to simplify its expression we normalise the root vectors:

$$e_\gamma = E_\gamma, \quad f_\gamma = F_\gamma / a_\gamma.$$

(The expression for f_γ is well-defined as $a_\gamma \neq 0$ [12, Eqs. (8.3)–(8.4)].) Then, we have

$$\tilde{R}_\gamma = \begin{cases} \sum_{k=0}^{\infty} \frac{(q - q^{-1})^k (e_\gamma \otimes f_\gamma)^k}{[k]_{q^{-2}}!}, & \text{if } [e_\gamma] = 0, \\ \sum_{k=0}^{\infty} \frac{(q^{-1} - q)^k (e_\gamma \otimes f_\gamma)^k}{[k]_{-q^{-1}}!}, & \text{if } [e_\gamma] = 1. \end{cases}$$

Define $\tilde{R} \in \overline{U}_q^+(\mathfrak{g}) \otimes \overline{U}_q^+(\mathfrak{g})$ by $\tilde{R} = \prod_{\gamma \in \phi} \tilde{R}_\gamma$ where the product is ordered using the same normal order $\mathcal{N}(\phi)$ that was used to define the root vectors in $U_q(\mathfrak{g})$; ie $\tilde{R} = \tilde{R}_{\gamma_1} \tilde{R}_{\gamma_2} \dots \tilde{R}_{\gamma_k}$ where $\mathcal{N}(\phi)$ is $\gamma_1 \prec \gamma_2 \prec \dots \prec \gamma_k$. Clearly, \tilde{R} is invertible as $\prod_{\gamma \in \phi} \tilde{R}_\gamma \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ is invertible and q is not a root of unity.

Lemma 5.1. Define an automorphism Ψ of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ by

$$\begin{aligned}\Psi(K_i^{\pm 1} \otimes 1) &= K_i^{\pm 1} \otimes 1, & \Psi(1 \otimes K_i^{\pm 1}) &= 1 \otimes K_i^{\pm 1}, \\ \Psi(e_i \otimes 1) &= e_i \otimes K_i^{-1}, & \Psi(1 \otimes e_i) &= K_i^{-1} \otimes e_i, \\ \Psi(f_i \otimes 1) &= f_i \otimes K_i, & \Psi(1 \otimes f_i) &= K_i \otimes f_i.\end{aligned}$$

The automorphism Ψ satisfies the following relations:

$$(i) \quad \tilde{R}\Delta(x) = \Psi(\Delta'(x)) \cdot \tilde{R}, \quad \text{for all } x \in U_q(\mathfrak{g}),$$

$$(ii) \quad (\Delta \otimes \text{id})\tilde{R} = \Psi_{23}(\tilde{R}_{13}) \cdot \tilde{R}_{23},$$

$$(iii) \quad (\text{id} \otimes \Delta)\tilde{R} = \Psi_{12}(\tilde{R}_{13}) \cdot \tilde{R}_{12},$$

where $\Psi_{12} = \Psi \otimes \text{id}$ and $\Psi_{23} = \text{id} \otimes \Psi$.

Proof. We prove (i) for each generator of $U_q(\mathfrak{g})$. Firstly, we wish to prove the following equations:

$$\tilde{R}(e_i \otimes K_i + 1 \otimes e_i) = (e_i \otimes K_i^{-1} + 1 \otimes e_i)\tilde{R}, \quad (13)$$

$$\tilde{R}(f_i \otimes 1 + K_i^{-1} \otimes f_i) = (f_i \otimes 1 + K_i \otimes f_i)\tilde{R}, \quad (14)$$

$$\tilde{R}(K_i^{\pm 1} \otimes K_i^{\pm 1}) = (K_i^{\pm 1} \otimes K_i^{\pm 1})\tilde{R}. \quad (15)$$

Eq. (15) is true by inspection and Eqs. (13)–(14) follow from the corresponding results in $U_h(\mathfrak{g})$ [12, Prop. 6.2]. The proof of (i) then follows from the definition of Ψ and the proofs of (ii) and (iii) follow similarly from [12]. \square

We now examine the usual approach used to create R -matrices for representations of a quantum algebra A . For each tensor product $W_1 \otimes W_2$ of finite dimensional integrable A -modules, an invertible element $\mathcal{E}_{W_1, W_2} \in \text{End}_{\mathbb{C}}(W_1 \otimes W_2)$ is constructed implementing the automorphism Ψ , in the sense that \mathcal{E}_{W_1, W_2} satisfies

$$\mathcal{E}_{W_1, W_2}^{-1} \cdot (\pi_{W_1} \otimes \pi_{W_2})(x) \cdot \mathcal{E}_{W_1, W_2} = (\pi_{W_1} \otimes \pi_{W_2})\Psi(x), \quad \forall x \in A \otimes A.$$

This \mathcal{E}_{W_1, W_2} is completely fixed by defining its action to be

$$\mathcal{E}_{W_1, W_2}(w_\lambda \otimes w_\mu) = q^{(\lambda, \mu)}(w_\lambda \otimes w_\mu),$$

on all weight vectors $w_\lambda \in W_1$, $w_\mu \in W_2$ with weights λ and μ , respectively [5, Prop. 10.1.19].

We could use the same method to construct R -matrices for representations of $U_q(\mathfrak{g})$ but we have found a more useful approach. Above, one needs to know the weight space decompositions of each of W_1 and W_2 before defining \mathcal{E}_{W_1, W_2} . With this knowledge, we can do something more interesting: instead of defining an element of $\text{End}_{\mathbb{C}}(W_1 \otimes W_2)$, we can define an element $E_{W_2} \in U_q(\mathfrak{g})$ with the property that $(\pi_{W_1} \otimes \pi_{W_2})E_{W_2} = \mathcal{E}_{W_1, W_2}$. A reason for doing this is that this allows us to define, for each finite dimensional tensorial irreducible representation V_λ of $U_q(\mathfrak{g})$, a class of even invertible elements of $\overline{U}_q^+(\mathfrak{g})$, each of which acts as a non-zero scalar multiple of the identity on each vector in V_λ . The specific non-zero scalar is $q^{-(\lambda, \lambda + 2\rho)}$. We define E_{W_2} for each tensor product of finite dimensional irreducible $U_q(\mathfrak{g})$ -modules following a related idea in [26].

For each $i = 1, \dots, n$, set $J_i = K_i K_{i+1} \cdots K_n$. The action of J_i on a weight vector w_ξ with weight $\xi = \sum_{j=1}^n \xi_j \epsilon_j \in H^*$ of a $U_q(\mathfrak{g})$ -module is

$$J_i w_\xi = q^{\xi_i} w_\xi.$$

Consider the weight space decomposition of a finite dimensional irreducible $U_q(\mathfrak{g})$ -module V_μ with integral dominant highest weight μ . The weight of the weight vector $w_\xi \in V_\mu$ is $\xi = \sum_{i=1}^n \xi_i \epsilon_i \in \bigoplus_{i=1}^n \mathbb{Z} \epsilon_i$. Now define

$$E_\mu = \prod_{a=1}^n \sum_{b=p}^s (J_a)^b \otimes P_a[b], \quad P_a[b] = \prod_{\substack{c=p \\ c \neq b}}^s \frac{J_a - q^c}{q^b - q^c}, \quad c \in \mathbb{Z}, \quad (16)$$

where p and s are any integers satisfying $p \leq s$ and the further condition that

(i) $J_i w_\xi = q^{\xi_i} w_\xi$ for some ξ_i satisfying $p \leq \xi_i \leq s$, for each weight vector $w_\xi \in V_\mu$.

Once we have any such p and s satisfying these conditions, we can use any p' and s' satisfying $p' \leq p$ and $s' \geq s$ in (16) instead of p and s , respectively.

The element E_μ is well-defined and invertible in $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, and for all weight vectors $v_{\lambda'} \in V_{\lambda}$ and $v_{\mu'} \in V_\mu$ we have

$$E_\mu(v_{\lambda'} \otimes v_{\mu'}) = q^{(\lambda', \mu')} (v_{\lambda'} \otimes v_{\mu'}), \quad (17)$$

where the weights of $v_{\lambda'}$ and $v_{\mu'}$ are λ' and μ' , respectively. The element E_μ is not a *universal* element in that it does not satisfy (17) for all representations of $U_q(\mathfrak{g})$; it would be useful if one could construct such a universal element.

Using this, we obtain R -matrices for tensor products of finite dimensional irreducible $U_q(\mathfrak{g})$ -modules in the following sense. Let V_λ and V_μ be irreducible $U_q(\mathfrak{g})$ -modules with integral dominant highest weights λ and μ , respectively. Then we have the following important theorem.

Theorem 5.2. *Define $R_\mu = E_\mu \tilde{R} \in \overline{U}_q^+(\mathfrak{g}) \otimes \overline{U}_q^+(\mathfrak{g})$ and $\mathcal{R}_{\lambda\mu} = (\pi_\lambda \otimes \pi_\mu) R_\mu$, then*

$$\mathcal{R}_{\lambda\mu} \cdot (\pi_\lambda \otimes \pi_\mu)(\Delta(x)) = (\pi_\lambda \otimes \pi_\mu)(\Delta'(x)) \cdot \mathcal{R}_{\lambda\mu}, \quad \forall x \in U_q(\mathfrak{g}). \quad (18)$$

Proof. This is similar to the proof of the corresponding theorem in quantum algebras [5, Prop. 10.1.19]. A direct calculation readily shows that

$$(\pi_\lambda \otimes \pi_\mu) \Psi(\Delta(x)) = (\pi_\lambda \otimes \pi_\mu)(E_\mu^{-1} \cdot \Delta(x) \cdot E_\mu), \quad \forall x \in U_q(\mathfrak{g}),$$

then by using $\tilde{R} \Delta(x) = \Psi(\Delta'(x)) \cdot \tilde{R}$ from Lemma 5.1, we have

$$(\pi_\lambda \otimes \pi_\mu)(R_\mu \Delta(x)) = (\pi_\lambda \otimes \pi_\mu)(\Delta'(x) R_\mu), \quad (19)$$

which is precisely Eq. (18). \square

We now determine some useful results involving R_μ .

Proposition 5.1. *The element R_μ has the following properties:*

$$(\epsilon \otimes \text{id}) R_\mu = (\text{id} \otimes \epsilon) R_\mu = 1, \quad (20)$$

$$(\pi_\lambda \otimes \pi_\mu)((S \otimes \text{id}) R_\mu) = (\pi_\lambda \otimes \pi_\mu) R_\mu^{-1}, \quad (\pi_\lambda \otimes \pi_\mu)((\text{id} \otimes S) R_\mu^{-1}) = (\pi_\lambda \otimes \pi_\mu) R_\mu, \quad (21)$$

$$(\pi_\lambda \otimes \pi_\mu)((S \otimes S) R_\mu) = (\pi_\lambda \otimes \pi_\mu) R_\mu. \quad (22)$$

Proof. Eq. (20) is proved by inspection. The proofs of (21)–(22) are straightforward and almost identical to the proofs of the corresponding equations in \mathbb{Z}_2 -graded quasitriangular Hopf algebras. \square

Let v_λ and v_ν be weight vectors of $U_q(\mathfrak{g})$ -modules with weights $\lambda, \nu \in \bigoplus_{i=1}^n \mathbb{Z} \epsilon_i$, respectively and let $v_{\mu'} \in V_\mu$ be a weight vector with weight μ' , then it can be easily shown that

$$[(\Delta \otimes 1) E_\mu](v_\lambda \otimes v_\nu \otimes v_{\mu'}) = q^{(\mu', \lambda + \nu)} (v_\lambda \otimes v_\nu \otimes v_{\mu'}),$$

$$[(1 \otimes \Delta) E_\mu](v_\lambda \otimes v_\nu \otimes v_{\mu'}) = q^{(\lambda, \nu + \mu')} (v_\lambda \otimes v_\nu \otimes v_{\mu'}),$$

where in $(1 \otimes \Delta) E_\mu$ we change the limits p and s if necessary.

We now consider analogues in $U_q(\mathfrak{g})$ of the equations $(\Delta \otimes 1) R = R_{13} R_{23}$ and $(1 \otimes \Delta) R = R_{13} R_{12}$ of a \mathbb{Z}_2 -graded quasitriangular Hopf algebra. By definition, we have $(\Delta \otimes 1) R_\mu = [(\Delta \otimes$

$1)E_\mu] \cdot \Psi_{23}(\tilde{R}_{13}) \cdot \tilde{R}_{23}$. Let V_λ and V_ν be finite dimensional irreducible $U_q(\mathfrak{g})$ -modules, then from the properties of E_μ we have

$$\begin{aligned} (\pi_\lambda \otimes \pi_\nu \otimes \pi_\mu)[(\Delta \otimes 1)R_\mu] &= (\pi_\lambda \otimes \pi_\nu \otimes \pi_\mu) \left[(\Delta \otimes 1)E_\mu \cdot (E_\mu^{23})^{-1} \tilde{R}_{13} E_\mu^{23} \tilde{R}_{23} \right] \\ &= (\pi_\lambda \otimes \pi_\nu \otimes \pi_\mu) \left[E_\mu^{13} \tilde{R}_{13} E_\mu^{23} \tilde{R}_{23} \right], \end{aligned} \quad (23)$$

where writing $E_\mu = \sum_t \alpha_t \otimes \beta_t$ we have $E_\mu^{13} = \sum_t \alpha_t \otimes \text{id} \otimes \beta_t$ and $E_\mu^{23} = \sum_t \text{id} \otimes \alpha_t \otimes \beta_t$. Note that (23) uses the result

$$(\pi_\lambda \otimes \pi_\nu \otimes \pi_\mu) [(\Delta \otimes 1)E_\mu \cdot (E_\mu^{23})^{-1}] = (\pi_\lambda \otimes \pi_\nu \otimes \pi_\mu) E_\mu^{13},$$

rather than an equality in $U_q(\mathfrak{g})^{\otimes 3}$. Similarly, we have

$$(1 \otimes \Delta)R_\mu = [(1 \otimes \Delta)E_\mu] \cdot \Psi_{12}(\tilde{R}_{13}) \cdot \tilde{R}_{12}, \quad \text{and}$$

$$(\pi_\lambda \otimes \pi_\nu \otimes \pi_\mu)[(1 \otimes \Delta)R_\mu] = (\pi_\lambda \otimes \pi_\nu \otimes \pi_\mu) \left[E_\mu^{13} \tilde{R}_{13} E_\nu^{12} \tilde{R}_{12} \right]. \quad (24)$$

Together with Theorem 5.2, this shows that R_μ satisfies the defining relations (5)–(7) of the universal R -matrix of a \mathbb{Z}_2 -graded quasitriangular Hopf algebra in *each triple tensor product* of finite dimensional irreducible $U_q(\mathfrak{g})$ representations, if we carefully choose the limits in the definition of E_μ , which can always be done. Furthermore, Eqs. (19) and (23)–(24) imply that

$$(\pi_\lambda \otimes \pi_\lambda \otimes \pi_\lambda) R_{12} R_{13} R_{23} = (\pi_\lambda \otimes \pi_\lambda \otimes \pi_\lambda) R_{23} R_{13} R_{12}, \quad (25)$$

where we have fixed $R = R_\lambda$.

For later use, note the following easily proved results. Define an automorphism $\Psi_m : U_q(\mathfrak{g})^{\otimes m} \rightarrow U_q(\mathfrak{g})^{\otimes m}$ generalising the automorphism $\Psi : U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ in Lemma 5.1 by

$$\begin{aligned} \Psi_m(1^{\otimes j} \otimes K_i^{\pm 1} \otimes 1^{\otimes(m-j-1)}) &= 1^{\otimes j} \otimes K_i^{\pm 1} \otimes 1^{\otimes(m-j-1)}, \\ \Psi_m(1^{\otimes j} \otimes e_i \otimes 1^{\otimes(m-j-1)}) &= (K_i^{-1})^{\otimes j} \otimes e_i \otimes (K_i^{-1})^{\otimes(m-j-1)}, \\ \Psi_m(1^{\otimes j} \otimes f_i \otimes 1^{\otimes(m-j-1)}) &= (K_i)^{\otimes j} \otimes f_i \otimes (K_i)^{\otimes(m-j-1)}, \end{aligned}$$

for each $1 \leq i \leq n$ and all $0 \leq j \leq m-1$. Then

$$(\Delta \otimes \text{id}^{\otimes t}) \Psi_{2,3,\dots,t+1}(\tilde{R}_{1(t+1)}) = \Psi_{2,3,\dots,t+2}(\tilde{R}_{1(t+2)}) \cdot \Psi_{3,4,\dots,t+2}(\tilde{R}_{2(t+2)}) \quad (26)$$

$$(\text{id}^{\otimes t} \otimes \Delta) \Psi_{1,2,\dots,t}(\tilde{R}_{1(t+1)}) = \Psi_{1,2,\dots,t+1}(\tilde{R}_{1(t+2)}) \cdot \Psi_{1,2,\dots,t}(\tilde{R}_{1(t+1)}) \quad (27)$$

where $\Psi_{k,\dots,m} = \text{id}^{\otimes(k-1)} \otimes \Psi_{m-k+1}$, $k \geq 2$, in (26) and $\Psi_{1,\dots,m} = \Psi_m \otimes \text{id}$ in (27). Then it may be easily shown that

$$\begin{aligned} (\pi^{\otimes t} \otimes \pi) (\Delta^{(t-1)} \otimes \text{id}) R &= (\pi^{\otimes t} \otimes \pi) R_{1(t+1)} R_{2(t+1)} \cdots R_{t(t+1)}, \\ (\pi \otimes \pi^{\otimes t}) (\text{id} \otimes \Delta^{(t-1)}) R &= (\pi \otimes \pi^{\otimes t}) R_{1(t+1)} R_{1t} \cdots R_{12}, \end{aligned} \quad (28)$$

where we fix $R = R_V$.

6 Two useful elements of $\overline{U}_q^+(\mathfrak{g})$

Define a set of elements $\{u_\lambda \in \overline{U}_q^+(\mathfrak{g}) \mid \lambda \in \mathcal{P}^+\}$ by

$$u_\lambda = \sum_t S(b_{\lambda_t}) a_{\lambda_t} (-1)^{[a_{\lambda_t}]},$$

where $R_\lambda = \sum_t a_{\lambda_t} \otimes b_{\lambda_t}$. The following lemma was proved in [4].

Lemma 6.1. *The element u_λ has the following properties:*

- (i) $\epsilon(u_\lambda) = 1$,
- (ii) $\pi_\lambda(S^2(x)u_\lambda) = \pi_\lambda(u_\lambda x)$, $\forall x \in U_q(\mathfrak{g})$.
- (iii) $\pi_\lambda(u_\lambda \tilde{u}_\lambda) = \pi_\lambda(1) = \pi_\lambda(\tilde{u}_\lambda u_\lambda)$, where \tilde{u}_λ is defined by $\tilde{u}_\lambda = \sum_s S^{-1}(d_{\lambda_s})c_{\lambda_s}(-1)^{[c_{\lambda_s}]}$, where $R_\lambda^{-1} = \sum_s c_{\lambda_s} \otimes d_{\lambda_s}$,
- (iv) $(\pi_\lambda \otimes \pi_\lambda)(\Delta(u_\lambda)) = (\pi_\lambda \otimes \pi_\lambda) \left[(u_\lambda \otimes u_\lambda)(R_\lambda^T R_\lambda)^{-1} \right]$.

Now define a set of elements $\{v_\lambda \in \overline{U}_q^+(\mathfrak{g}) \mid \lambda \in \mathcal{P}^+\}$ by

$$v_\lambda = u_\lambda K_{2\rho}^{-1}. \quad (29)$$

Lemma 6.2. *The element v_λ has the following properties:*

$$\begin{aligned} \epsilon(v_\lambda) &= 1, & \pi_\lambda(v_\lambda x) &= \pi_\lambda(x v_\lambda), & \forall x \in U_q(\mathfrak{g}), \\ (\pi_\lambda \otimes \pi_\lambda)\Delta(v_\lambda) &= (\pi_\lambda \otimes \pi_\lambda) \left[(v_\lambda \otimes v_\lambda)(R_\lambda^T R_\lambda)^{-1} \right]. \end{aligned} \quad (30)$$

Proof. The proofs of $\epsilon(v_\lambda) = 1$ and (30) follow from the definition of v_λ . To prove the remaining relation, note that $S^2(e_i) = K_i e_i K_i^{-1} = K_{2\rho} e_i K_{2\rho}^{-1}$, $S^2(f_i) = K_i f_i K_i^{-1} = K_{2\rho} f_i K_{2\rho}^{-1}$, and $S^2(K_i^{\pm 1}) = K_{2\rho} K_i^{\pm 1} K_{2\rho}^{-1}$. As S^2 is a homomorphism we have $S^2(x) = K_{2\rho} x K_{2\rho}^{-1}$ for all $x \in U_q(\mathfrak{g})$ and then

$$\begin{aligned} \pi_\lambda(v_\lambda x v_\lambda^{-1}) &= \pi_\lambda(u_\lambda K_{2\rho}^{-1} x K_{2\rho} u_\lambda^{-1}) \\ &= \pi_\lambda(u_\lambda S^{-2}(x) u_\lambda^{-1}) = \pi_\lambda(S^2(S^{-2}(x))) = \pi_\lambda(x), \end{aligned}$$

completing the proof. \square

Lemma 6.3. *The element v_λ acts on each vector in the irreducible $U_q(\mathfrak{g})$ -module V_λ with highest weight $\lambda \in \mathcal{P}^+$ as the multiplication by the scalar $q^{-(\lambda+2\rho, \lambda)}$.*

Proof. Note that v_λ is even and that $\pi_\lambda(v_\lambda) \in \text{End}_{U_q(\mathfrak{g})}(V_\lambda)$. Write $R_\lambda = E_\lambda \tilde{R}$ where $\tilde{R} = \sum_{t=0}^\infty a_t \otimes b_t$, $a_t \in U_q(\mathfrak{b}_+)$, $b_t \in U_q(\mathfrak{b}_-)$ and $a_0 = b_0 = 1$, then

$$\pi_\lambda(v_\lambda) = \pi_\lambda \left(\sum_{t=0}^\infty S(b_t) E a_t K_{2\rho}^{-1} (-1)^{[a_t]} \right),$$

where E is an even element of $U_q(\mathfrak{g})$ satisfying $E w_\xi = q^{-(\xi, \xi)} w_\xi$ for each weight vector $w_\xi \in V_\lambda$ of weight $\xi \in \bigoplus_{i=1}^n \mathbb{Z} \epsilon_i$. Let w_λ be a non-zero highest weight vector of V_λ , then $a_t w_\lambda = 0$ for all $t > 0$, yielding

$$v_\lambda \cdot w_\lambda = E K_{2\rho}^{-1} w_\lambda = q^{-(\lambda+2\rho, \lambda)} w_\lambda.$$

As it is true that V_λ is irreducible, $\pi_\lambda(v_\lambda) \in \text{End}_{U_q(\mathfrak{g})}(V_\lambda)$ and $\pi_\lambda(v_\lambda)$ is a homogeneous map of degree zero, it follows that v_λ acts on each weight vector $w \in V_\lambda$ as the multiplication by the claimed scalar from Schur's lemma. \square

We may denote $q^{-(\lambda+2\rho, \lambda)}$ by $\chi_\lambda(v_\lambda)$. Note that v_μ may act on each weight vector $w \in V_\lambda$ as the multiplication by the scalar $q^{-(\lambda+2\rho, \lambda)}$ even if $\mu \neq \lambda$, and in this case we also write $\chi_\lambda(v_\mu)$ to denote $q^{-(\lambda+2\rho, \lambda)}$.

Following [28], we define the quantum superdimension of the finite dimensional $U_q(\mathfrak{g})$ -module W to be

$$sdim_q(W) = \text{str}_q(\text{id}_W).$$

The following lemma, first stated in [27], was proved in [4] drawing on [10, 11].

Lemma 6.4. *Let V_λ be a finite dimensional irreducible $U_q(\mathfrak{g})$ -module with integral dominant highest weight λ . The quantum superdimension of V_λ is*

$$sdim_q(V_\lambda) = (-1)^{[\lambda]} q^{-(\lambda, 2\rho)} \prod_{\alpha \in \bar{\Phi}_0^+} \left(\frac{q^{2(\lambda+\rho, \alpha)} - 1}{q^{2(\rho, \alpha)} - 1} \right) \prod_{\beta \in \Phi_1^+} \left(\frac{q^{2(\lambda+\rho, \beta)} + 1}{q^{2(\rho, \beta)} + 1} \right), \quad (31)$$

where $[\lambda]$ is the grading of the highest weight vector of V_λ .

It is easy to calculate that the quantum superdimension of the fundamental $(2n+1)$ -dimensional $U_q(\mathfrak{osp}(1|2n))$ -module V is

$$sdim_q(V) = 1 - \frac{q^{2n} - q^{-2n}}{q - q^{-1}}, \quad (32)$$

where we recall that the grading of the highest weight vector of V is odd.

7 Spectral decomposition of $\check{\mathcal{R}}_{V,V}$

Let V_λ and V_μ be finite dimensional irreducible $U_q(\mathfrak{g})$ -modules with integral dominant highest weights λ and μ , respectively. Let R_μ be as in Theorem 5.2 and define $\check{\mathcal{R}}_{V_\lambda, V_\mu} \in \text{Hom}_{\mathbb{C}}(V_\lambda \otimes V_\mu, V_\mu \otimes V_\lambda)$ by

$$\check{\mathcal{R}}_{V_\lambda, V_\mu}(v_\lambda \otimes v_\mu) = P \circ (R_\mu(v_\lambda \otimes v_\mu)), \quad (33)$$

where $v_\lambda \in V_\lambda$ and $v_\mu \in V_\mu$ are weight vectors and P is the graded permutation operator. A standard argument proves the following lemma:

Lemma 7.1. *Let V_λ be an irreducible $U_q(\mathfrak{g})$ -module with integral dominant highest weight λ , then $\check{\mathcal{R}}_{V_\lambda, V_\lambda} \in \text{End}_{U_q(\mathfrak{g})}(V_\lambda \otimes V_\lambda)$.*

For $n = 1$, $V \otimes V$ decomposes into a direct sum of irreducible $U_q(\mathfrak{g})$ -modules:

$$V \otimes V = V_{2\epsilon_1} \oplus V_{\epsilon_1} \oplus V_0, \quad (34)$$

and for $n \geq 2$, we have

$$V \otimes V = V_{2\epsilon_1} \oplus V_{\epsilon_1 + \epsilon_2} \oplus V_0. \quad (35)$$

Lemma 7.2. *Let $n \geq 2$ and let $\{P[\mu] \in \text{End}_{U_q(\mathfrak{g})}(V \otimes V) \mid \mu = 2\epsilon_1, \epsilon_1 + \epsilon_2, 0\}$ be a set of even $U_q(\mathfrak{g})$ -linear maps: $P[\mu] : V \otimes V \rightarrow V \otimes V$, where the image of $P[\mu]$ is V_μ and the maps satisfy $(P[\mu])^2 = P[\mu]$ and $P[\mu]P[\nu] = \delta_{\mu\nu}P[\mu]$. Then there is a spectral decomposition of $\check{\mathcal{R}}_{V,V}$:*

$$\check{\mathcal{R}}_{V,V} = -qP[2\epsilon_1] + q^{-1}P[\epsilon_1 + \epsilon_2] + q^{-2n}P[0].$$

Proof. As $\check{\mathcal{R}}_{V,V} \in \text{End}_{U_q(\mathfrak{g})}(V \otimes V)$, we can write

$$\check{\mathcal{R}}_{V,V} = \beta_{2\epsilon_1}P[2\epsilon_1] + \beta_{\epsilon_1 + \epsilon_2}P[\epsilon_1 + \epsilon_2] + \beta_0P[0],$$

for some set of constants $\{\beta_\mu \in \mathbb{C} \mid \mu = 2\epsilon_1, \epsilon_1 + \epsilon_2, 0\}$, where β_μ is the scalar action of $\check{\mathcal{R}}_{V,V}$ on the irreducible $U_q(\mathfrak{g})$ -submodule $V_\mu \subset V \otimes V$. We explicitly calculate each β_μ using R_V .

Let $\{v_i \mid -n \leq i \leq n\}$ be the basis of weight vectors of V given in Lemma 4.3. The highest weight vector of $V_{2\epsilon_1}$ is $w_{2\epsilon_1} = v_1 \otimes v_1$, the highest weight vector of $V_{\epsilon_1 + \epsilon_2}$ is $w_{\epsilon_1 + \epsilon_2} = v_1 \otimes v_2 - q^{-1}v_2 \otimes v_1$ and the highest weight vector of the trivial module $V_0 \subset V \otimes V$ is $w_0 = \sum_{i=-n}^n c_i v_i \otimes v_{-i}$, where $\{c_i \in \mathbb{C} \mid -n \leq i \leq n\}$ is a set of non-zero constants inductively defined by

$$\begin{aligned} c_n &= -c_0, & c_{-n} &= q^{-1}c_0, \\ c_{n-1} &= -qc_n, & c_{-(n-1)} &= -q^{-1}c_{-n}, \\ c_i &= -qc_{i+1}, & c_{-i} &= -q^{-1}c_{-(i+1)}, \end{aligned}$$

where $i = 1, 2, \dots, n-2$ and we fix $c_0 \neq 0$.

To study the action of $\tilde{\mathcal{R}}_{V,V}$ on the highest weight vectors $w_{2\epsilon_1}, w_{\epsilon_1+\epsilon_2}$ and w_0 , we make some observations about $(\pi \otimes \pi)\tilde{R}$. From the weight space decomposition of V , we have

$$\begin{aligned}\pi(f_{\epsilon_i})^3 &= \pi(e_{\epsilon_i})^3 = 0, & \text{for all } i = 1, \dots, n, \\ \pi(f_\gamma)^2 &= \pi(e_\gamma)^2 = 0, & \text{for all } \gamma \in \phi \text{ where } \gamma \neq \epsilon_i,\end{aligned}$$

and thus

$$(\pi \otimes \pi)\tilde{R} = (\pi \otimes \pi) \prod_{\gamma \in \phi} \tilde{R}_\gamma^V, \quad (36)$$

where

$$\begin{aligned}\tilde{R}_\gamma^V &= \begin{cases} \sum_{k=0}^2 \frac{(q^{-1}-q)^k (e_\gamma \otimes f_\gamma)^k}{[k]^{-q^{-1}}!}, & \text{if } \gamma = \epsilon_i, \\ \sum_{k=0}^1 \frac{(q-q^{-1})^k (e_\gamma \otimes f_\gamma)^k}{[k]^{q^{-2}}!}, & \text{if } \gamma \neq \epsilon_i. \end{cases} \\ &= \begin{cases} 1 \otimes 1 + (q^{-1}-q)(e_\gamma \otimes f_\gamma) + \frac{(q^{-1}-q)^2 (e_\gamma \otimes f_\gamma)^2}{(1-q^{-1})}, & \text{if } \gamma = \epsilon_i, \\ 1 \otimes 1 + (q-q^{-1})(e_\gamma \otimes f_\gamma), & \text{if } \gamma \neq \epsilon_i, \end{cases}\end{aligned}$$

and where the product in (36) is ordered using the same normal ordering $\mathcal{N}(\phi)$ used to construct the root vectors, ie we fix $\prod_{\gamma \in \phi} \tilde{R}_\gamma^V = \tilde{R}_{\gamma_1}^V \tilde{R}_{\gamma_2}^V \cdots \tilde{R}_{\gamma_k}^V$ where $\mathcal{N}(\phi) = \gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_k$. The expression for $(\pi \otimes \pi)\tilde{R}$ in (36) readily assists the use of $(\pi \otimes \pi)R_V$ and $\tilde{\mathcal{R}}_{V,V}$ in calculations.

Using this, we have $\tilde{\mathcal{R}}_{V,V}(w_{2\epsilon_1}) = -qw_{2\epsilon_1}$ and $\tilde{\mathcal{R}}_{V,V}(w_{\epsilon_1+\epsilon_2}) = q^{-1}w_{\epsilon_1+\epsilon_2}$. Calculating β_0 is more difficult: note that

$$\tilde{\mathcal{R}}_{V,V} \left(c_{-1}v_{-1} \otimes v_1 + \sum_{\substack{i=-n \\ i \neq -1}}^n c_i v_i \otimes v_{-i} \right) = -q^{-1}c_{-1}v_1 \otimes v_{-1} + \sum_{\substack{j=-n \\ j \neq -1}}^n c'_j v_{-j} \otimes v_j,$$

for some set of non-zero constants $\{c'_j \in \mathbb{C} \mid -n \leq j \leq n, j \neq -1\}$. Recall that $\tilde{\mathcal{R}}_{V,V}(w_0) = \beta_0 w_0$, so we obtain β_0 by comparing $-q^{-1}c_{-1}$ and c_1 . Now $c_{-1} = (-1)^{n-1}q^{-n}c_0$ and $c_1 = (-1)^n q^{n-1}c_0$, thus $\beta_0 = q^{-2n}$. \square

Lemma 7.3. *Let $n = 1$ and let $\{P[\mu] \in \text{End}_{U_q(\mathfrak{g})}(V \otimes V) \mid \mu = 2\epsilon_1, \epsilon_1, 0\}$ be a set of even $U_q(\mathfrak{g})$ -linear maps: $P[\mu] : V \otimes V \rightarrow V \otimes V$, where the image of $P[\mu]$ is V_μ and the maps satisfy $(P[\mu])^2 = P[\mu]$ and $P[\mu]P[\nu] = \delta_{\mu\nu}P[\mu]$. Then there is a spectral decomposition of $\tilde{\mathcal{R}}_{V,V}$:*

$$\tilde{\mathcal{R}}_{V,V} = -qP[2\epsilon_1] + q^{-1}P[\epsilon_1] + q^{-2}P[0].$$

Proof. The proof is almost identical to the proof of Lemma 7.2 except for the following minor difference. The highest weight vector of V_{ϵ_1} in the decomposition of $V \otimes V$ in (34) is $w_{\epsilon_1} = v_1 \otimes v_0 + q^{-1}v_0 \otimes v_1$. To complete the proof we note that $\tilde{\mathcal{R}}_{V,V}(w_{\epsilon_1}) = q^{-1}w_{\epsilon_1}$. \square

Corollary 7.1. *For each $n = 1, 2, \dots$, $\tilde{\mathcal{R}}_{V,V}$ satisfies*

$$(\tilde{\mathcal{R}}_{V,V} + q)(\tilde{\mathcal{R}}_{V,V} - q^{-1})(\tilde{\mathcal{R}}_{V,V} - q^{-2n}) = 0. \quad (37)$$

8 A representation of the Birman-Wenzl-Murakami algebra $BW_t(-q^{2n}, q)$

In this section we recall the Birman-Wenzl-Murakami algebra $BW_t(r, q)$ from [3, 24] and define a representation of $BW_t(-q^{2n}, q)$ in a subalgebra \mathcal{C}_t of the centraliser algebra $\text{End}_{U_q(\mathfrak{g})}(V^{\otimes t})$.

Definition 8.1. Define \mathcal{C}_t to be the subalgebra of $\text{End}_{U_q(\mathfrak{g})}(V^{\otimes t})$ generated by the elements

$$\{\check{R}_i^{\pm 1} \in \text{End}_{U_q(\mathfrak{g})}(V^{\otimes t}) \mid 1 \leq i \leq t-1\}, \quad \text{where}$$

$$\check{R}_i = \text{id}^{\otimes(i-1)} \otimes \check{\mathcal{R}}_{V,V} \otimes \text{id}^{\otimes(t-(i+1))} \in \text{End}_{U_q(\mathfrak{g})}(V^{\otimes t}). \quad (38)$$

Let us investigate \mathcal{C}_t . Let $\{v_i \mid -n \leq i \leq n\}$ be the basis of weight vectors of V given in Lemma 4.3 and let $\{v_i^* \mid -n \leq i \leq n\}$ be a basis of V^* such that $\langle v_i^*, v_j \rangle = \delta_{ij}$ and $[v_i^*] = [v_i]$; then

$$av_i = \sum_j \langle v_j^*, av_i \rangle v_j, \quad av_i^* = \sum_j \langle av_i^*, v_j \rangle v_j^*, \quad \forall a \in U_q(\mathfrak{g}).$$

Define $\check{e} \in \text{End}_{\mathbb{C}}(V \otimes V^*)$ by

$$\check{e}(x \otimes y^*) = (-1)^{[y^*][x]} \langle y^*, v^{-1}u x \rangle \sum_{i=-n}^n v_i \otimes v_i^*,$$

where v and u are the elements $v_{\epsilon_1}, u_{\epsilon_1} \in \overline{U}_q^+(\mathfrak{g})$ respectively.

Lemma 8.1. The map \check{e} satisfies

- (i) $(\check{e})^2 = \text{sdim}_q(V)\check{e}$,
- (ii) $a\check{e} = \epsilon(a)\check{e}$, $\forall a \in U_q(\mathfrak{g})$,
- (iii) $\check{e}a = \epsilon(a)\check{e}$, $\forall a \in U_q(\mathfrak{g})$,
- (iv) $\check{e}_2\check{R}_1\check{e}_2 = q^{2n}\check{e}_2$, where

$$\begin{aligned} \check{e}_2 &= \text{id}_V \otimes \check{e} : V \otimes V \otimes V^* \rightarrow V \otimes V \otimes V^*, \\ \check{R}_1 &= \check{\mathcal{R}}_{V,V} \otimes \text{id}_{V^*} : V \otimes V \otimes V^* \rightarrow V \otimes V \otimes V^*. \end{aligned}$$

Proof. (i)

$$\begin{aligned} (\check{e})^2(x \otimes y^*) &= (-1)^{[y^*][x]} \langle y^*, v^{-1}ux \rangle \sum_i (-1)^{[v_i]} \langle v_i^*, v^{-1}u v_i \rangle \sum_j v_j \otimes v_j^* \\ &= \text{sdim}_q(V) (-1)^{[y^*][x]} \langle y^*, v^{-1}u x \rangle \sum_j v_j \otimes v_j^* = \text{sdim}_q(V) \check{e}(x \otimes y^*). \end{aligned}$$

(ii) By definition, $a\check{e} = a_{V \otimes V^*} \circ \check{e}$; we calculate that

$$\begin{aligned} a\check{e}(x \otimes y^*) &= (-1)^{[y^*][x]} \langle y^*, v^{-1}ux \rangle \sum_{(a), i, j, k} \langle v_j^*, a_{(1)}v_i \rangle \langle v_i^*, S(a_{(2)})v_k \rangle v_j \otimes v_k^* \\ &= (-1)^{[y^*][x]} \langle y^*, v^{-1}ux \rangle \sum_{(a), j, k} \langle v_j^*, a_{(1)}S(a_{(2)})v_k \rangle v_j \otimes v_k^* \\ &= \epsilon(a) (-1)^{[y^*][x]} \langle y^*, v^{-1}ux \rangle \sum_k v_k \otimes v_k^* = \epsilon(a) \check{e}(x \otimes y^*). \end{aligned}$$

Similar calculations prove (iii) and (iv) (see [15] for the corresponding calculations in ungraded quasitriangular Hopf algebras). \square

Define $\hat{e} \in \text{End}_{\mathbb{C}}(V^* \otimes V)$ by

$$\hat{e}(x^* \otimes y) = \langle x^*, y \rangle \sum_{i=-n}^n (-1)^{[v_i]} v_i^* \otimes v u^{-1} v_i,$$

where v and u are fixed to be v_{ϵ_1} and u_{ϵ_1} , respectively.

Lemma 8.2. *The map \hat{e} satisfies*

- (i) $(\hat{e})^2 = \text{sdim}_q(V)\hat{e}$,
- (ii) $a\hat{e} = \epsilon(a)\hat{e}, \quad \forall a \in U_q(\mathfrak{g})$,
- (iii) $\hat{e}a = \epsilon(a)\hat{e}, \quad \forall a \in U_q(\mathfrak{g})$,
- (iv) $\hat{e}_2\check{R}_1^{-1}\hat{e}_2 = q^{-2n}\hat{e}_2$ where $\hat{e}_2 = \text{id}_{V^*} \otimes \hat{e} : V^* \otimes V^* \otimes V \rightarrow V^* \otimes V^* \otimes V$.

Proof. The proofs of (i)–(iv) are similar to the proofs of (i)–(iv) in Lemma 8.1. \square

Remark 8.1. *The maps \check{e} and \hat{e} are $U_q(\mathfrak{g})$ -invariant maps onto one-dimensional $U_q(\mathfrak{g})$ -submodules in $V \otimes V^*$ and $V^* \otimes V$, respectively.*

Recall that $V \otimes V$ has the decomposition into irreducible $U_q(\mathfrak{g})$ -modules given in (34)–(35) and that there exists an even $U_q(\mathfrak{g})$ -invariant map $P[0] : V \otimes V \rightarrow V \otimes V$ the image of which is $V_0 \subset V \otimes V$, defined in Lemmas 7.2–7.3. Recall that $V^* \cong V$ and define

$$E = (\text{id} \otimes T^{-1}) \circ \check{e} \circ (\text{id} \otimes T) = (T^{-1} \otimes \text{id}) \circ \hat{e} \circ (T \otimes \text{id}) = \text{sdim}_q(V)P[0],$$

where T is the isomorphism $T : V \rightarrow V^*$ given in (9). Furthermore, define the elements

$$E_i = \text{id}^{\otimes(i-1)} \otimes E \otimes \text{id}^{\otimes(t-(i+1))} \in \text{End}_{U_q(\mathfrak{g})}(V^{\otimes t}), \quad i = 1, 2, \dots, t-1.$$

Lemma 8.3. *The elements $\check{R}_i, E_i \in \text{End}_{U_q(\mathfrak{g})}(V^{\otimes t})$ satisfy the relations*

- (i) $\check{R}_i\check{R}_{i+1}\check{R}_i = \check{R}_{i+1}\check{R}_i\check{R}_{i+1}, \quad 1 \leq i \leq t-2$,
- (ii) $\check{R}_i\check{R}_j = \check{R}_j\check{R}_i, \quad |i-j| > 1$,
- (iii) $(\check{R}_i + q)(\check{R}_i - q^{-1})(\check{R}_i - q^{-2n}) = 0, \quad 1 \leq i \leq t-1$,
- (iv) $-\check{R}_i + \check{R}_i^{-1} = (q - q^{-1})(1 - E_i)$,
- (v) $E_i\check{R}_{i-1}^{\pm 1}E_i = q^{\pm 2n}E_i$,
- (vi) $E_i\check{R}_i^{\pm 1} = \check{R}_i^{\pm 1}E_i = q^{\mp 2n}E_i, \quad 1 \leq i \leq t-1$.

Proof. The proofs of (i) and (ii) are standard arguments. The proof of (iii) follows from Corollary 7.1. The proof of (v) follows from Lemmas 8.1–8.2. The proofs of (iv) and (vi) follow from the definition of E_i , Eq. (32) and the fact that \check{R}_1 acts on $V_0 \subset V \otimes V$ as $\check{R}_1 w = q^{-2n}w$ for all $w \in V_0$. \square

We now give the definition of the Birman-Wenzl-Murakami algebra $BW_t(r, q)$ from [21]. Let r and q be non-zero complex constants and let $t \geq 2$ be an integer. The Birman-Wenzl-Murakami algebra $BW_t(r, q)$ is the algebra over \mathbb{C} generated by the invertible elements $\{g_i \mid 1 \leq i \leq t-1\}$ subject to the relations

$$\begin{aligned} g_i g_j &= g_j g_i, & |i-j| > 1, \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, & 1 \leq i \leq t-2, \\ e_i g_i &= r^{-1} e_i, & 1 \leq i \leq t-1, \\ e_i g_{i-1}^{\pm 1} e_i &= r^{\pm 1} e_i, & 1 \leq i \leq t-1, \end{aligned}$$

where e_i is defined by

$$(q - q^{-1})(1 - e_i) = g_i - g_i^{-1}, \quad 1 \leq i \leq t-1.$$

It can be shown that each g_i also satisfies

$$(g_i - r^{-1})(g_i + q^{-1})(g_i - q) = 0.$$

From Lemma 8.3 we have the following:

Lemma 8.4. *Let $q \in \mathbb{C}$ be non-zero and not a root of unity. The algebra homomorphism $\Upsilon : BW_t(-q^{2n}, q) \rightarrow \mathcal{C}_t$ defined by*

$$\Upsilon : g_i \mapsto -\tilde{R}_i$$

yields a representation of $BW_t(-q^{2n}, q)$ in \mathcal{C}_t .

9 Bratteli diagrams and path algebras

9.1 Bratteli diagrams

To proceed further with the study of the Birman-Wenzl-Murakami algebra we consider the notions of *Bratteli diagrams* and *Path algebras for Bratteli diagrams*, both of which we take from [15]. (The reader is also referred to [6, Chap. 2]).

A Bratteli diagram is an undirected graph that encodes information about a sequence $\mathbb{C} \cong A_0 \subset A_1 \subset A_2 \subset \dots$ of inclusions of finite dimensional semisimple algebras [21]. The properties of a Bratteli diagram, graph-theoretically, are that:

- (i) its vertices are elements of certain sets \tilde{A}_t , $t \in \mathbb{Z}_+$, and
- (ii) if $n(a, b) \in \mathbb{Z}_+$ denotes the number of edges between the vertices a and b , then $n(a, b) = 0$ for any vertices $a \in \tilde{A}_i$ and $b \in \tilde{A}_j$ where $|i - j| \neq 1$.

Assume that \tilde{A}_0 consists of a unique vertex that we denote by \emptyset . We call the elements of \tilde{A}_i *shapes* and say that \tilde{A}_i is the set of shapes on the i^{th} level of the Bratteli diagram. If $\lambda \in \tilde{A}_i$ is connected to $\mu \in \tilde{A}_{i+1}$, we write $\lambda \leq \mu$.

A *multiplicity free Bratteli diagram* is a Bratteli diagram in which any two vertices are connected by at most one edge. All Bratteli diagrams considered in this paper are multiplicity free.

Let A be a Bratteli diagram and let $\lambda \in \tilde{A}_i$ and $\mu \in \tilde{A}_j$ for some $0 \leq i < j$. We define a *path from λ to μ* to be a sequence of shapes

$$P = (s_i, s_{i+1}, \dots, s_j),$$

where $\lambda = s_i \leq s_{i+1} \leq \dots \leq s_{j-1} \leq s_j = \mu$ and s_k is a shape on the k^{th} level of A for each k .

Given a path $T = (\lambda, \dots, \mu)$ from λ to μ and a path $S = (\mu, \dots, \nu)$ from μ to ν , we define the concatenation of T and S to be the path from λ to ν defined by

$$T \circ S = (\lambda, \dots, \mu, \dots, \nu).$$

We define a tableau T of shape λ to be a path from $\emptyset \in \tilde{A}_0$ to λ and we write $shp(T) = \lambda$. We say that the length of T is t if there are $t + 1$ shapes in the tableau.

9.2 Path algebras related to Bratteli diagrams

We now define the concept of a *Path algebra* for a Bratteli diagram A . For each $t \in \mathbb{Z}_+$, let \mathcal{T}^t be the set of tableaux of length t in A and let $\Omega^t \subset \mathcal{T}^t \times \mathcal{T}^t$ be the set of pairs (S, T) of tableaux where $shp(S) = shp(T)$, that is both S and T end in the same shape. Let us further define an algebra A_t over \mathbb{C} generated by $\{E_{ST} \mid (S, T) \in \Omega^t\}$, where the algebra multiplication is defined by

$$E_{ST}E_{PQ} = \delta_{TP}E_{SQ}. \quad (39)$$

Any set of elements of an associative algebra satisfying (39) are called matrix units; matrix units also figure later in this paper. Note that $A_0 \cong \mathbb{C}$. Each element $a \in A_t$ can be written in the form

$$a = \sum_{(S, T) \in \Omega^t} a_{ST} E_{ST}, \quad a_{ST} \in \mathbb{C}.$$

We refer to the collection of algebras A_t , $t \in \mathbb{Z}_+$, as the *tower of path algebras corresponding to the Bratteli diagram A* .

Each of the algebras A_t is isomorphic to a direct sum of matrix algebras. The irreducible representations of A_t are indexed by the elements of \tilde{A}_t , which is the set of shapes on the t^{th} level of A . Let \mathcal{T}^λ denote the set of tableaux of shape λ , then the cardinality d_λ of $\mathcal{T}^\lambda \cap \mathcal{T}^t$ is equal to the dimension of the irreducible A_t -module indexed by $\lambda \in \tilde{A}_t$. We record this in the formula

$$A_t \cong \bigoplus_{\lambda \in \tilde{A}_t} M_{d_\lambda}(\mathbb{C}),$$

where $M_d(\mathbb{C})$ denotes the algebra of $d \times d$ matrices with complex entries.

We now define some useful sets. Let \mathcal{T}_λ^μ be the set of paths in A from the shape λ to the shape μ and let \mathcal{T}_r^t be the set of paths starting on the r^{th} level of A and going down to the t^{th} level. Furthermore, let \mathcal{T}_λ^t be the set of paths in A from the shape λ to any shape on the t^{th} level of A .

We also define $\Omega_\lambda^\mu \subset \mathcal{T}_\lambda^\mu \times \mathcal{T}_\lambda^\mu$ to be the set of pairs (S, T) of paths $S, T \in \mathcal{T}_\lambda^\mu$ and $\Omega_r^t \subset \mathcal{T}_r^t \times \mathcal{T}_r^t$ to be the set of pairs (S, T) of paths where in both situations we have $shp(S) = shp(T)$.

We define the inclusion of path algebras $A_r \subseteq A_t$ for $0 \leq r < t$ as follows: for each pair $(P, Q) \in \Omega^r$ we fix $E_{PQ} \in A_t$ by

$$E_{PQ} = \sum_{T \in \mathcal{T}_\lambda^t \cap \mathcal{T}_r^t} E_{P \circ T, Q \circ T}, \quad \text{where } \lambda = shp(P) = shp(Q).$$

In particular, we have $A_s \subseteq A_{s+1}$ for each $s \in \mathbb{Z}_+$.

Let $\lambda \in \tilde{A}_t$ and let \mathcal{V}_λ be an irreducible representation of A_t indexed by λ . The restriction of \mathcal{V}_λ to the subalgebra $A_{t-1} \subseteq A_t$ decomposes into irreducible representations of A_{t-1} according to

$$\mathcal{V}_\lambda \downarrow_{A_{t-1}}^{A_t} \cong \bigoplus_{\mu \in \lambda^-} \mathcal{V}_\mu, \quad \text{where } \lambda^- = \{\nu \in \tilde{A}_{t-1} \mid \nu \leq \lambda\}.$$

This decomposition is multiplicity free as the Bratteli diagram A is multiplicity free.

For each $r \in \mathbb{Z}_+$ satisfying $r < t$, the *centraliser of A_r contained in A_t* is defined to be

$$\mathcal{L}(A_r \subseteq A_t) = \{a \in A_t \mid ab = ba, \forall b \in A_r\}.$$

Let now (S, T) be a pair of paths each starting on the r^{th} level of A at the shape λ and ending on the t^{th} level of A at the shape μ . For each such pair we define $E_{ST} \in A_t$ by

$$E_{ST} = \sum_{P \in \mathcal{T}^\lambda \cap \mathcal{T}^r} E_{P \circ S, P \circ T},$$

which we can think of as the sum of all pairs of paths 'ending' in (S, T) . We then have the following lemma, stated in [15, Prop. (1.4)] and proved in [6, Sect. 2.3].

Lemma 9.1. *A basis of $\mathcal{L}(A_r \subseteq A_t)$ is given by the elements*

$$\left\{ E_{ST} \mid (S, T) \in \Omega_\lambda^\mu \cap \Omega_r^t, \lambda \in \tilde{A}_r, \mu \in \tilde{A}_t \right\}.$$

9.3 Centraliser algebras

Let U be a \mathbb{Z}_2 -graded Hopf algebra over \mathbb{C} . Let V be a finite dimensional U -module with the property that $V^{\otimes t}$ is completely reducible for each $t \in \mathbb{Z}_+$. We now define the concepts of a *Bratteli diagram for tensor powers of V* and the *Bratteli diagram for $V^{\otimes t}$* ; the purpose of this subsection is to show that the centraliser \mathcal{L}_t of U in $\text{End}_{\mathbb{C}}(V^{\otimes t})$ defined by $\mathcal{L}_t = \text{End}_U(V^{\otimes t})$ is isomorphic to the path algebra A_t of the Bratteli diagram for $V^{\otimes t}$.

In this subsection we regard all modules as being graded. By convention $V^{\otimes 0} \cong \mathbb{C}$ and thus $\mathcal{L}_0 = \mathbb{C}$. If V is an irreducible U -module then $\mathcal{L}_1 \cong \mathbb{C}$ by Schur's lemma. For all $0 \leq r < t$ we

define the inclusion $\mathcal{L}_r \subseteq \mathcal{L}_t$ by $a \mapsto a \otimes \text{id}^{\otimes(t-r)}$ for all $a \in \mathcal{L}_r$. Now \mathcal{L}_t acts on $V^{\otimes t}$ in the obvious way. Since U and \mathcal{L}_t commute, $V^{\otimes t}$ has a natural $\mathcal{L}_t \otimes U$ -module structure.

Let $\{\Lambda_\lambda \mid \lambda \in I\}$ be the set of non-isomorphic finite dimensional irreducible U -modules. Then by the double centraliser theorem there exists a finite subset $\tilde{\mathcal{L}}_t$ of I such that

$$V^{\otimes t} \cong \bigoplus_{\lambda \in \tilde{\mathcal{L}}_t} \mathcal{L}^\lambda \otimes \Lambda_\lambda,$$

where each \mathcal{L}^λ is an irreducible \mathcal{L}_t -module such that $\mathcal{L}^\lambda \not\cong \mathcal{L}^\mu$ if $\lambda \neq \mu$.

We now assume that V is an irreducible U -module and continue to assume that all tensor powers of V are completely reducible. We will consider the Bratteli diagram for tensor powers of V . Let $\lambda \in \tilde{\mathcal{L}}_t$ for some t . Then we have the decomposition

$$\Lambda_\lambda \otimes V = \bigoplus_{\mu \in \tilde{\mathcal{L}}_{t+1}} (\Lambda_\mu)^{\oplus n_\lambda(\mu)}, \quad n_\lambda(\mu) \in \mathbb{Z}_+, \quad (40)$$

of $\Lambda_\lambda \otimes V$ into a direct sum of irreducible U -modules. The non-negative integer $n_\lambda(\mu)$ is the multiplicity of Λ_μ in the decomposition. We say that the decomposition of $\Lambda_\lambda \otimes V$ is *multiplicity free* if $n_\lambda(\mu) \leq 1$ for all $\mu \in \tilde{\mathcal{L}}_{t+1}$.

The *branching rule for inclusion* $\mathcal{L}_t \subseteq \mathcal{L}_{t+1}$ describes the decomposition of the \mathcal{L}_{t+1} -module \mathcal{L}^ν into \mathcal{L}_t -modules

$$\mathcal{L}^\nu = \bigoplus_{\lambda \in \tilde{\mathcal{L}}_t} (\mathcal{L}^\lambda)^{\oplus n_\lambda(\nu)}, \quad n_\lambda(\nu) \in \mathbb{Z}_+. \quad (41)$$

Note that the positive integers $n_\lambda(\nu)$ appearing in (40) and (41) are the same [15].

The *Bratteli diagram for tensor powers of V* is defined as follows: for each $t \in \mathbb{Z}_+$ fix the vertices on the t^{th} level of the Bratteli diagram to be the elements of $\tilde{\mathcal{L}}_t$. Then a vertex $\lambda \in \tilde{\mathcal{L}}_t$ is connected to a vertex $\mu \in \tilde{\mathcal{L}}_{t+1}$ by $n_\lambda(\mu)$ edges.

For a fixed t , the *Bratteli diagram for $V^{\otimes t}$* is an undirected graph with vertices given by the elements of $\bigcup_{i=0}^t \tilde{\mathcal{L}}_i$, and the edges are such that a vertex $\lambda \in \tilde{\mathcal{L}}_i$ is connected to a vertex $\mu \in \tilde{\mathcal{L}}_{i+1}$ by $n_\lambda(\mu)$ edges for each $0 \leq i \leq t-1$.

Let V be a finite dimensional irreducible U -module with the property that for every irreducible U -module W , the decomposition of the tensor product $W \otimes V$ is multiplicity free. In this case, we say that tensoring by V is multiplicity free. We will show that the centraliser algebra $\mathcal{L}_t = \text{End}_U(V^{\otimes t})$ is isomorphic to the path algebra A_t associated with the Bratteli diagram for $V^{\otimes t}$.

We construct an algebra isomorphism $A_t \rightarrow \mathcal{L}_t$ inductively. Assume that there is an identification of \mathcal{L}_t with the path algebra A_t for some $t \geq 0$, so that

$$V^{\otimes t} = \bigoplus_{\lambda \in \tilde{\mathcal{L}}_t} \left(\bigoplus_{T \in \mathcal{T}^\lambda \cap \mathcal{T}^t} E_{TT} V^{\otimes t} \right)$$

is a decomposition of $V^{\otimes t}$ into irreducible U -modules Λ_λ where the U -submodule $E_{TT} V^{\otimes t}$ is isomorphic to Λ_λ given $\text{shp}(T) = \lambda$. The map E_{TT} is a U -invariant map from $V^{\otimes t}$ onto a U -submodule isomorphic to Λ_λ .

Let $T = (\emptyset, s_1, \dots, \lambda) \in \mathcal{T}^\lambda$ be a tableau of length t and let $E_{TT} V^{\otimes t} \cong \Lambda_\lambda$ for some $\lambda \in \tilde{\mathcal{L}}_t$. As tensoring by V is multiplicity free, the decomposition

$$(E_{TT} V^{\otimes t}) \otimes V = \bigoplus_{\substack{\nu \in \tilde{\mathcal{L}}_{t+1} \\ \lambda \leq \nu}} V_{T \circ \nu}, \quad (42)$$

is multiplicity free and thus unique, where $T \circ \nu$ is the tableau

$$T \circ \nu = (\emptyset, s_1, \dots, \lambda, \nu), \quad \lambda \leq \nu,$$

and $V_{T \circ \nu} \cong \Lambda_\nu$.

The next step is to identify $E_{T \circ \nu, T \circ \nu}$ with the unique U -invariant projection operator mapping $(E_{TT} V^{\otimes t}) \otimes V$ onto $V_{T \circ \nu}$. This way we identify each element E_{SS} of the path algebra A_{t+1} , where $S \in \mathcal{T}^{t+1}$, with an element of \mathcal{L}_{t+1} . Thus we have the decomposition

$$V^{\otimes(t+1)} = \bigoplus_{\nu \in \tilde{\mathcal{L}}_{t+1}} \left(\bigoplus_{S \in \mathcal{T}^\nu \cap \mathcal{T}^{t+1}} E_{SS} V^{\otimes(t+1)} \right),$$

of $V^{\otimes(t+1)}$ into irreducible U -modules $E_{SS} V^{\otimes(t+1)} = V_S \cong \Lambda_\nu$, where $\nu \in \tilde{\mathcal{L}}_{t+1}$ and $S \in \mathcal{T}^\nu \cap \mathcal{T}^{t+1}$.

We now identify the other elements in the basis $\{E_{PQ} \in A_{t+1} \mid (P, Q) \in \Omega^{t+1}\}$ with elements of \mathcal{L}_{t+1} . For each pair of paths $(P, Q) \in \Omega^{t+1}$ we choose non-zero elements

$$E_{PQ} \in E_{PP} \mathcal{L}_{t+1} E_{QQ}, \quad E_{QP} \in E_{QQ} \mathcal{L}_{t+1} E_{PP},$$

normalised in such a way that $E_{PQ} E_{QP} = E_{PP}$ and $E_{QP} E_{PQ} = E_{QQ}$. Thus there is an algebra isomorphism $A_{t+1} \rightarrow \mathcal{L}_{t+1}$.

We then have the following theorem.

Theorem 9.1. *Let V be a finite dimensional irreducible U -module such that $V^{\otimes t}$ is completely reducible for each $t \in \mathbb{Z}_+$ and such that tensoring by V is multiplicity free. Then for any $t \in \mathbb{Z}_+$, the centraliser algebra $\mathcal{L}_t = \text{End}_U(V^{\otimes t})$ is isomorphic to the path algebra A_t corresponding to the Bratteli diagram for $V^{\otimes t}$.*

10 Projections from $V^{\otimes t}$ onto its irreducible $U_q(\mathfrak{g})$ -submodules

Recall from Lemma 4.2 that all tensor products of irreducible $U_q(\mathfrak{g})$ -modules with integral dominant highest weights at generic q are completely reducible. In this section we define projections from $V^{\otimes t}$ onto all the irreducible $U_q(\mathfrak{g})$ -submodules $V_\lambda \subset V^{\otimes t}$, $\lambda \in \mathcal{P}^+$, using elements of \mathcal{C}_t . Recall from Section 4 that \mathcal{P}^+ is the set of integral dominant weights. No substantially new results appear in this section, however, we are not aware of this specific formulation of the projections in the literature.

Let V_μ be a finite dimensional irreducible $U_q(\mathfrak{g})$ -module with highest weight $\mu \in \mathcal{P}^+$. Since each weight space of V is one-dimensional, $V_\mu \otimes V$ is multiplicity free, and from Lemma 4.2 we know the highest weights of the irreducible $U_q(\mathfrak{g})$ -submodules in $V_\mu \otimes V$.

Definition 10.1. *We define $\mathcal{P}_\mu^+ \subset \mathcal{P}^+$ to be the set such that for each $\lambda \in \mathcal{P}_\mu^+$, V_λ appears in $V_\mu \otimes V$ as an irreducible $U_q(\mathfrak{g})$ -submodule.*

Now each $\lambda \in \mathcal{P}_\mu^+$ can only have one of the following three forms: $\mu, \mu + \epsilon_i, \mu - \epsilon_i$ for some i . Thus

$$\mathcal{P}_\mu^+ \subseteq \mathcal{P}_\mu^0 = \{\mu, \mu \pm \epsilon_i \in \mathcal{P}^+ \mid 1 \leq i \leq n\}.$$

Definition 10.2. *Let V_μ be an irreducible $U_q(\mathfrak{g})$ -module with highest weight $\mu \in \mathcal{P}^+$, then $V_\mu \otimes V = \bigoplus_{\lambda \in \mathcal{P}_\mu^+} V_\lambda$. Let $\{p_\mu[\lambda] \in \text{End}_{U_q(\mathfrak{g})}(V_\mu \otimes V) \mid \lambda \in \mathcal{P}_\mu^+\}$ be a set of even maps*

$$p_\mu[\lambda] : V_\mu \otimes V \rightarrow V_\mu \otimes V$$

such that

(i) the image of $p_\mu[\lambda]$ is V_λ ,

(ii) $(p_\mu[\lambda])^2 = p_\mu[\lambda]$,

(iii) $p_\mu[\lambda] \cdot p_\mu[\nu] = \delta_{\lambda\nu} p_\mu[\lambda]$.

We call each such $p_\mu[\lambda]$ a projection.

Recall that for each integral dominant λ , there exists an element $v_\lambda \in \overline{U}_q^+(\mathfrak{g})$ defined in (29) that acts on each vector in the finite dimensional irreducible $U_q(\mathfrak{g})$ -module V_λ as the multiplication by the scalar $q^{-(\lambda+2\rho, \lambda)}$.

For each $\mu \in \mathcal{P}^+$ and each $\lambda \in \mathcal{P}_\mu^+$, define $p_\mu[\lambda] \in \text{End}_{U_q(\mathfrak{g})}(V_\mu \otimes V)$ by

$$p_\mu[\lambda] = (\pi_\mu \otimes \pi) \left(\prod_{\substack{\nu \in \mathcal{P}_\mu^+ \\ \nu \neq \lambda}} \frac{\Delta(v_\xi) - q^{-(\nu+2\rho, \nu)}}{q^{-(\lambda+2\rho, \lambda)} - q^{-(\nu+2\rho, \nu)}} \right), \quad (43)$$

where v_ξ is the element $v_\lambda \in \overline{U}_q^+(\mathfrak{g})$ with $\lambda = \xi$, for some integral dominant ξ which is chosen so that v_ξ acts as the multiplication by the scalar $q^{-(\nu+2\rho, \nu)}$ on each vector in the irreducible $U_q(\mathfrak{g})$ -module V_ν , for each $\nu \in \mathcal{P}_\mu^+$. For each integral dominant μ there always exists at least one such ξ . To see this, all we need is some E_ξ given by Eq. (16): $E_\xi = \prod_{a=1}^n \sum_{b=p}^s (J_a)^b \otimes P_a[b]$, such

that the element $E = \prod_{a=1}^n \sum_{b=p}^s P_a[b] (J_a)^{-b}$ acts as the multiplication by the scalar $q^{-(\zeta, \zeta)}$ on each weight vector $w_\zeta \in V_\nu \subseteq V_\mu \otimes V$, where w_ζ has the weight ζ , and this is true for each $\nu \in \mathcal{P}_\mu^+$.

The element E has this action whenever s and $|p|$ are sufficiently large enough, and so all we need do is to choose some ξ for which this is true. To do this, let I_ν be the set of distinct weights of the weight vectors of V_ν for each $\nu \in \mathcal{P}_\mu^+$, then $(\zeta_\nu, \epsilon_i) \in \mathbb{Z}$ for each weight $\zeta_\nu \in I_\nu$ and each $i = 1, \dots, n$. Let

$$m = \max \left\{ |(\zeta_\nu, \epsilon_i)| \in \mathbb{Z}_+ \mid \zeta_\nu \in I_\nu, \nu \in \mathcal{P}_\mu^+, i = 1, \dots, n \right\},$$

then fixing $\xi = \sum_{i=1}^n m \epsilon_i$ yields elements E_ξ and E with the desired properties.

Note that $(\pi_\mu \otimes \pi) \Delta(v_\xi)$ in (43) is diagonalisable as $V_\mu \otimes V$ is completely reducible and $\Delta(v_\xi)$ acts on each irreducible $U_q(\mathfrak{g})$ -submodule $V_\nu \subset V_\mu \otimes V$ as the multiplication by the scalar $q^{-(\nu+2\rho, \nu)}$.

Lemma 10.1. *The maps $p_\mu[\lambda]$ are well-defined and satisfy*

- (i) $(p_\mu[\lambda])^2 = p_\mu[\lambda]$,
- (ii) $p_\mu[\lambda] \cdot p_\mu[\nu] = \delta_{\lambda\nu} p_\mu[\lambda]$,
- (iii) $\sum_{\lambda \in \mathcal{P}_\mu^+} p_\mu[\lambda] = \text{id}_{V_\mu \otimes V}$.

Proof. If α and β are the highest weights of irreducible $U_q(\mathfrak{g})$ -submodules in $V_\mu \otimes V$, then $(\alpha + 2\rho, \alpha) = (\beta + 2\rho, \beta)$ implies that $\alpha = \beta$. Then $p_\mu[\lambda]$ is well defined as tensoring by V is multiplicity free and q is not a root of unity. The proof of (i) follows from the result that $(p_\mu[\lambda])^2 (V_\mu \otimes V) = p_\mu[\lambda] (V_\mu \otimes V)$. For (ii) the case $\lambda = \nu$ reduces to (i), and for $\lambda \neq \nu$ we have

$$\begin{aligned} & p_\mu[\lambda] \cdot p_\mu[\nu] \\ &= (\pi_\mu \otimes \pi) \left(\prod_{\substack{\lambda' \in \mathcal{P}_\mu^+ \\ \lambda' \neq \lambda}} \frac{\Delta(v_\xi) - q^{-(\lambda'+2\rho, \lambda')}}{q^{-(\lambda+2\rho, \lambda)} - q^{-(\lambda'+2\rho, \lambda')}} \prod_{\substack{\nu' \in \mathcal{P}_\mu^+ \\ \nu' \neq \nu}} \frac{\Delta(v_\xi) - q^{-(\nu'+2\rho, \nu')}}{q^{-(\nu+2\rho, \nu)} - q^{-(\nu'+2\rho, \nu')}} \right) = 0. \end{aligned}$$

$$(iii) \quad \sum_{\lambda \in \mathcal{P}_\mu^+} p_\mu[\lambda] (V_\mu \otimes V) = \bigoplus_{\lambda \in \mathcal{P}_\mu^+} V_\lambda = V_\mu \otimes V.$$

□

Note that $(\pi_\mu \otimes \pi) \prod_{\lambda \in \mathcal{P}_\mu^+} (\Delta(v_\xi) - q^{-(\lambda+2\rho, \lambda)}) = 0$.

We introduce some notation. Let \mathcal{T}^t be the set of tableaux of length t derived from the Bratteli diagram for $V^{\otimes t}$. Let

$$i^t = (0, s_1, \dots, s_t) \in \mathcal{T}^t.$$

We write $\lambda_i^t = i^t$ where $\lambda = s_t$.

Let $i^t \in \mathcal{T}^t$ and $s_j, s_{j+1} \in i^t$. Define a map

$$p_{s_j}^{t-(j+1)}[s_{j+1}] : (V_{s_j} \otimes V) \otimes V^{\otimes(t-(j+1))} \rightarrow V_{s_{j+1}} \otimes V^{\otimes(t-(j+1))}$$

by

$$p_{s_j}^{t-(j+1)}[s_{j+1}] = p_{s_j}[s_{j+1}] \otimes \text{id}^{\otimes(t-(j+1))}.$$

Lemma 10.2. *The map $p_{s_j}^{t-(j+1)}[s_{j+1}]$ satisfies*

$$(i) \quad (p_{s_j}^{t-(j+1)}[s_{j+1}])^2 = p_{s_j}^{t-(j+1)}[s_{j+1}],$$

$$(ii) \quad p_{s_j}^{t-(j+1)}[s_{j+1}] \cdot p_{s_j}^{t-(j+1)}[r_{j+1}] = \delta_{s_{j+1}, r_{j+1}} p_{s_j}^{t-(j+1)}[s_{j+1}],$$

$$(iii) \quad \sum_{s_{j+1} \in \mathcal{P}_{s_j}^+} p_{s_j}^{t-(j+1)}[s_{j+1}] = \text{id}_{V_{s_j} \otimes V^{\otimes(t-j)}}.$$

Proof. The proofs of parts (i) and (ii) follow from Lemma 10.1 (i) and (ii), respectively. The proof of (iii) follows from Lemma 10.1 (iii): explicitly, we have

$$\sum_{s_{j+1} \in \mathcal{P}_{s_j}^+} p_{s_j}^{t-(j+1)}[s_{j+1}] \cdot (V_{s_j} \otimes V) \otimes V^{\otimes(t-(j+1))} = V_{s_j} \otimes V \otimes V^{\otimes t-(j+1)}.$$

□

Definition 10.3. *Let $\tilde{p}_i^t[\lambda] \in \text{End}_{U_q(\mathfrak{g})}(V^{\otimes t})$ be a map $\tilde{p}_i^t[\lambda] : V^{\otimes t} \rightarrow V_\lambda \subset V^{\otimes t}$ defined by*

$$\tilde{p}_i^t[\lambda] = p_{s_{t-1}}^0[\lambda] p_{s_{t-2}}^1[s_{t-1}] \cdots p_{\epsilon_1}^{t-2}[s_2],$$

where $\lambda_i^t \in \mathcal{T}^t$. We say that $\tilde{p}_i^t[\lambda]$ projects from $V^{\otimes t}$ onto V_λ by the path $\lambda_i^t \in \mathcal{T}^t$ and we call $\tilde{p}_i^t[\lambda]$ a path projection of length t .

Lemma 10.3. *The map $\tilde{p}_i^t[\lambda]$ satisfies*

$$(i) \quad (\tilde{p}_i^t[\lambda])^2 = \tilde{p}_i^t[\lambda],$$

$$(ii) \quad \tilde{p}_i^t[\lambda] \cdot \tilde{p}_j^t[\lambda] = \begin{cases} 0, & \text{if } i^t \neq j^t, \\ \tilde{p}_i^t[\lambda], & \text{if } i^t = j^t, \end{cases}$$

$$(iii) \quad \tilde{p}_i^t[\lambda] \cdot \tilde{p}_j^t[\mu] = 0 \text{ if } \lambda \neq \mu.$$

Furthermore, the map $P_t = \sum_{i^t \in \mathcal{T}^t} \tilde{p}_i^t[\lambda]$ is the identity on $V^{\otimes t}$.

Proof.

(i) This follows from Lemma 10.2 (i).

(ii) For $i^t = j^t$ the case reduces to (i), let $i^t \neq j^t$ where

$$\begin{aligned} i^t &= (0, s_1, s_2, \dots, s_{k-1}, s_k, r_{k+1}, \dots, r_{t-2}, r_{t-1}, \lambda), \\ j^t &= (0, s_1, s_2, \dots, s_{k-1}, s_k, s_{k+1}, \dots, s_{t-2}, s_{t-1}, \lambda). \end{aligned}$$

Now $i^t, j^t \in \mathcal{T}^t$ and $r_{k+1} \neq s_{k+1}$ for some $2 \leq k+1 \leq t$, then

$$\begin{aligned} \tilde{p}_i^t[\lambda] \cdot \tilde{p}_j^t[\lambda] &= p_{r_{t-1}}^0[\lambda] p_{r_{t-2}}^1[r_{t-1}] \cdots p_{s_k}^{t-k-1}[r_{k+1}] p_{s_{k-1}}^{t-k}[s_k] p_{s_{k-2}}^{t-k+1}[s_{k-1}] \cdots p_{s_1}^{t-2}[s_2] \\ &\quad \times p_{s_{t-1}}^0[\lambda] p_{s_{t-2}}^1[s_{t-1}] \cdots p_{s_k}^{t-k-1}[s_{k+1}] p_{s_{k-1}}^{t-k}[s_k] p_{s_{k-2}}^{t-k+1}[s_{k-1}] \cdots p_{s_1}^{t-2}[s_2] \\ &= p_{r_{t-1}}^0[\lambda] p_{s_{t-1}}^0[\lambda] p_{r_{t-2}}^1[r_{t-1}] p_{s_{t-2}}^1[s_{t-1}] \cdots p_{s_k}^{t-k-1}[r_{k+1}] p_{s_k}^{t-k-1}[s_{k+1}] \\ &\quad \times p_{s_{k-1}}^{t-k}[s_k] p_{s_{k-1}}^{t-k}[s_k] p_{s_{k-2}}^{t-k+1}[s_{k-1}] p_{s_{k-2}}^{t-k+1}[s_{k-1}] \cdots p_{s_1}^{t-2}[s_2] p_{s_1}^{t-2}[s_2] \\ &= 0, \end{aligned}$$

$$\text{as } p_{s_k}^{t-k-1}[r_{k+1}] \cdot p_{s_k}^{t-k-1}[s_{k+1}] = 0.$$

(iii) Assume that

$$\begin{aligned} i^t &= (0, s_1, s_2, \dots, s_{k-1}, s_k, r_{k+1}, \dots, r_{t-2}, r_{t-1}, \lambda), \\ j^t &= (0, s_1, s_2, \dots, s_{k-1}, s_k, s_{k+1}, \dots, s_{t-2}, s_{t-1}, \mu), \end{aligned}$$

where $r_{k+1} \neq s_{k+1}$ for some $2 \leq k+1 \leq t$. The calculations are similar to those of (ii) and we have $\tilde{p}_i^t[\lambda] \cdot \tilde{p}_j^t[\mu] = 0$.

The last claim follows inductively from the result that $\sum_{\lambda \in \mathcal{P}_\mu^+} p_\mu[\lambda] = \text{id}_{V_\mu \otimes V}$. \square

Recall that \mathcal{C}_t is the algebra over \mathbb{C} generated by the elements

$$\{\check{R}_i^{\pm 1} \in \text{End}_{U_q(\mathfrak{g})}(V^{\otimes t}) \mid 1 \leq i \leq t-1\}.$$

Proposition 10.1. *For each path $\lambda_i^t \in \mathcal{T}^t$, $\tilde{p}_i^t[\lambda] \in \mathcal{C}_t$.*

Proof. We prove the proposition inductively. Firstly, for some appropriately chosen integral dominant weight ξ ,

$$(\pi \otimes \pi) \Delta(v_\xi) = (\pi \otimes \pi) \left[(v_\xi \otimes v_\xi) (R_\xi^T R_\xi)^{-1} \right] = q^{-2(\epsilon_1 + 2\rho, \epsilon_1)} \check{\mathcal{R}}^{-2} \in \mathcal{C}_2.$$

Now assume that $\tilde{p}_i^{(t-1)}[\mu] \in \mathcal{C}_{(t-1)}$ where $\tilde{p}_i^{(t-1)}[\mu]$ is a path projection $\tilde{p}_i^{(t-1)}[\mu] : V^{\otimes(t-1)} \rightarrow V_\mu$ and V_μ is an irreducible $U_q(\mathfrak{g})$ -submodule of $V^{\otimes(t-1)}$. We will show that $(\pi_\mu \otimes \pi) \Delta(v_\zeta)$ is an element of \mathcal{C}_t for some appropriately chosen ζ . Let ζ be an integral dominant weight such that the element $v_\zeta \in \overline{U}_q^+(\mathfrak{g})$ acts as the multiplication by the scalar $q^{-(\lambda + 2\rho, \lambda)}$ on each vector in the irreducible $U_q(\mathfrak{g})$ -submodule $V_\lambda \subset V_\mu \otimes V$ for each $\lambda \in \mathcal{P}_\mu^+$. Now

$$\begin{aligned} &(\pi_\mu \otimes \pi) \Delta(v_\zeta) \\ &= (\pi_\mu \otimes \pi) \left[(v_\zeta \otimes v_\zeta) (R_\zeta^T R_\zeta)^{-1} \right] \\ &= q^{-(\mu + 2\rho, \mu) - (\epsilon_1 + 2\rho, \epsilon_1)} (\tilde{p}_i^{(t-1)}[\mu] \otimes \text{id}) \left(\pi^{\otimes(t-1)} \otimes \pi \right) \left(\Delta^{(t-2)} \otimes \text{id} \right) (R_\zeta^T R_\zeta)^{-1} \\ &= q^{-(\mu + 2\rho, \mu) - (\epsilon_1 + 2\rho, \epsilon_1)} (\tilde{p}_i^{(t-1)}[\mu] \otimes \text{id}) \check{R}_{t-1}^{-1} \check{R}_{t-2}^{-1} \cdots \check{R}_1^{-1} \check{R}_1^{-1} \cdots \check{R}_{t-2}^{-1} \check{R}_{t-1}^{-1}, \end{aligned}$$

where we have used the following result from (28): (writing $R = R_\zeta$)

$$\left(\pi^{\otimes(t-1)} \otimes \pi \right) \left(\Delta^{(t-2)} \otimes \text{id} \right) R = \left(\pi^{\otimes(t-1)} \otimes \pi \right) R_{1t} R_{2t} \cdots R_{(t-1)t}.$$

\square

11 Matrix units for \mathcal{C}_t

It is clear that the Bratteli diagram for $V^{\otimes t}$ is multiplicity free, as tensoring by the fundamental $U_q(\mathfrak{g})$ -module V is multiplicity free. It follows then from Theorem 9.1 that the centraliser algebra $\mathcal{L}_t = \text{End}_{U_q(\mathfrak{g})}(V^{\otimes t})$ is isomorphic to the path algebra A_t obtained from the Bratteli diagram for $V^{\otimes t}$. Clearly, we have the inclusion $\mathcal{C}_t \subseteq \mathcal{L}_t$. The aim of this section is to show that \mathcal{C}_t and \mathcal{L}_t are in fact equal:

Theorem 11.1. *The centraliser algebra $\mathcal{L}_t = \text{End}_{U_q(\mathfrak{g})}(V^{\otimes t})$ is generated by the elements*

$$\{\check{R}_i^{\pm 1} \in \text{End}_{U_q(\mathfrak{g})}(V^{\otimes t}) \mid i = 1, 2, \dots, t-1\}.$$

To prove this theorem we firstly partition the matrix units in A_t into two groups: the *projectors* $\{E_{SS} \in A_t \mid (S, S) \in \Omega^t\}$ and the *intertwiners* $\{E_{ST} \in A_t \mid (S, T) \in \Omega^t, S \neq T\}$ and we use an invertible homomorphism to map matrix units in A_t to matrix units in \mathcal{C}_t .

Recall that $V^{\otimes t}$ is completely reducible. Each matrix unit in \mathcal{C}_t corresponding to a projector in A_t projects down from $V^{\otimes t}$ onto an irreducible $U_q(\mathfrak{g})$ -submodule $V_\lambda \subseteq V^{\otimes t}$. Each matrix unit in \mathcal{C}_t corresponding to an intertwiner in A_t maps between isomorphic irreducible $U_q(\mathfrak{g})$ -submodules of $V^{\otimes t}$.

Recall that the homomorphism $\Upsilon : g_i \mapsto -\check{R}_i$ given in Lemma 8.4 yields a representation of $BW_t(-q^{2n}, q)$ in \mathcal{C}_t . In Subsection 11.1 we will write down the matrix units in a semisimple quotient of $BW_t(-q^{2n}, q)$ that map via Υ onto the projectors and intertwiners in \mathcal{C}_t . We will do this for the intertwiners, but we choose to define the projectors more straightforwardly using our previous work.

The projections E_{SS} that project down from $V^{\otimes t}$ onto irreducible $U_q(\mathfrak{g})$ -submodules $V_{shp(S)} \subseteq V^{\otimes t}$ that we defined in Section 10, are elements of \mathcal{C}_t , and satisfy $(E_{SS})^2 = E_{SS}$ and $\sum_{S \in \mathcal{T}^t} E_{SS} = \text{id}_{V^{\otimes t}}$. We fix the projectors in \mathcal{C}_t by mapping the projector $E_{SS} \in A_t$ to $\tilde{p}_i^t[\lambda] \in \mathcal{C}_t$, where $\lambda_i^t = S \in \mathcal{T}^t$ is a path of length t : $E_{SS} \mapsto \tilde{p}_i^t[\lambda]$.

All we need do now is construct the matrix units in \mathcal{C}_t corresponding to the intertwiners in A_t . We denote the matrix unit in \mathcal{C}_t corresponding to $E_{MP} \in A_t$ also by E_{MP} .

11.1 Matrix units in $BW_t(-q^{2n}, q)$

In this subsection, we say that an algebra B is semisimple if it is isomorphic to a direct sum of matrix algebras, ie $B \cong \bigoplus_{i \in I} M_{b_i}(\mathbb{C})$, where $M_{b_i}(\mathbb{C})$ is the algebra of $b_i \times b_i$ matrices with complex entries. The algebra $BW_t(-q^{2n}, q)$ is not semisimple at generic q [24, Cor. 5.6] but Ram and Wenzl have constructed matrix units for the semisimple Birman-Wenzl-Murakami algebra BW_t defined over $\mathbb{C}(r, q)$ (the field of rational functions in r and q) for indeterminates r and q [21]. By replacing the indeterminates r and q with the complex numbers $-q^{2n}$ and q , respectively, we obtain matrix units in some appropriate semisimple quotient of $BW_t(-q^{2n}, q)$. By then applying the homomorphism Υ to these matrix units, we obtain matrix units in \mathcal{C}_t .

Before doing this, let us recall the definition of a Young diagram and discuss a relation between certain Young diagrams and integral dominant highest weights of irreducible $U_q(\mathfrak{osp}(1|2n))$ -modules. For each non-negative integer m , there exists a Young diagram for each partition of m . Let $m = m_1 + m_2 + \dots + m_l$ be a partition of m , where $m_i - m_{i+1} \in \mathbb{Z}_+$ for each $i = 1, 2, \dots, l-1$ and $m_l \in \mathbb{Z}_+$. The Young diagram representing this partition is a collection of m boxes arranged in l left-aligned rows where the i^{th} row from the top contains exactly m_i boxes. For $m \geq 1$, let $c_i, i = 1, 2, \dots, m_1$, be the number of boxes in the i^{th} column from the left in the Young diagram, then $c_i - c_{i+1} \in \mathbb{Z}_+$ for each $i = 1, 2, \dots, m_1 - 1$ and $c_{m_1} \in \{1, 2, \dots, l\}$.

Recall that an integral dominant highest weight λ of an irreducible $U_q(\mathfrak{g})$ -module V_λ has the form $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \mathcal{P}^+$ where $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$ for each $i = 1, 2, \dots, n-1$ and $\lambda_n \in \mathbb{Z}_+$. We can use a Young diagram to label λ : this Young diagram consists of $\sum_{i=1}^n \lambda_i$ boxes arranged in n left-aligned rows, where the i^{th} row contains exactly λ_i boxes.

Let μ be a Young diagram containing no more than n rows of boxes and let μ_i be the number of boxes in the i^{th} row from the top for each $i = 1, 2, \dots, n$. We can use the Young diagram μ

to label the integral dominant highest weight $\sum_{i=1}^n \mu_i \epsilon_i \in \mathcal{P}^+$ of an irreducible representation of $U_q(\mathfrak{osp}(1|2n))$.

11.2 The algebra BW_t

The Birman-Wenzl-Murakami algebra BW_t , with r and q indeterminates, is semisimple [24, Thm. 3.5]. To discuss the structure of BW_t , we introduce the Young lattice. For later purposes we note that BW_t is equipped with a functional $\text{tr} : BW_t \rightarrow \mathbb{C}(r, q)$ which satisfies, amongst other relations [24, Lem. 3.4 (d)],

$$\text{tr}(a\chi b) = \text{tr}(\chi)\text{tr}(ab), \quad \forall a, b \in BW_{t-1}, \quad \chi \in \{g_{t-1}, e_{t-1}\}, \quad (44)$$

where we regard each element of BW_{t-1} as an element of BW_t under the canonical inclusion.

Remark 11.1. *The functional tr in (44) can be used to define the Kauffman link invariant following [24, p. 404].*

The Young lattice is the following infinite graph [24, Sec. 1]. The vertices of the Young lattice are the Young diagrams; the vertices are grouped into levels so that each Young diagram with exactly t boxes labels a vertex on the t^{th} level of the Young lattice. The edges of the Young lattice are completely determined as follows: a vertex λ on the t^{th} level is connected to a vertex μ on the $(t+1)^{\text{st}}$ level by one edge if and only if λ and μ differ by exactly one box. We show the Young lattice up to the 4^{th} level in Figure 1, where the circle represents the Young diagram with no boxes. We say that the level containing the Young diagram with no boxes is the 0^{th} level. Note that the Young lattice is (apart from the 0^{th} level) identical to the Bratteli diagram for the sequence of inclusions of group algebras of the symmetric group: $\mathbb{C}S_1 \subset \mathbb{C}S_2 \subset \mathbb{C}S_3 \subset \dots$.

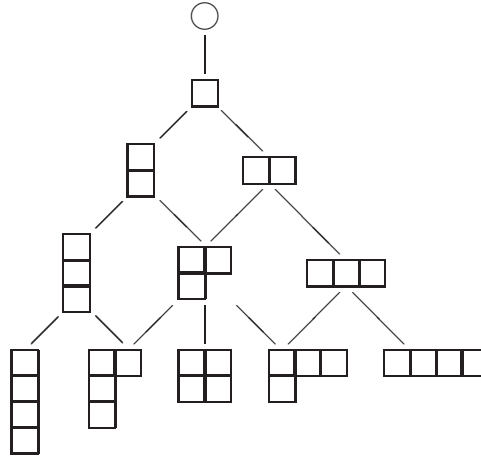


Figure 1: The Young lattice up to the 4^{th} level

For each t , let Y_t be the set of vertices on the t^{th} level of the Young lattice and define

$$\Gamma_t = \bigcup_{\substack{k \in \mathbb{Z}_+ \\ t-2k \geq 0}} Y_{t-2k}.$$

Then BW_t is isomorphic to a direct sum of matrix algebras [24, Thm. 3.5]:

$$BW_t \cong \bigoplus_{\mu \in \Gamma_t} M_{b_\mu}(\mathbb{C}).$$

Ram and Wenzl defined matrix units for BW_t [21] which we explicitly write down below for completeness.

To label the matrix units of BW_t , we need to discuss the Bratteli diagram of BW_t , which is the following graph. The vertices of the Bratteli diagram for BW_t are divided into levels; for each $s = 0, 1, \dots, t$, the vertices on the s^{th} level are precisely the elements of Γ_s . The edges are as follows: a vertex μ on the s^{th} level is connected to a vertex λ on the $(s+1)^{st}$ level if and only if μ and λ differ by exactly one box. We show the Bratteli diagram for BW_t up to the 4^{th} level in Figure 11.2.

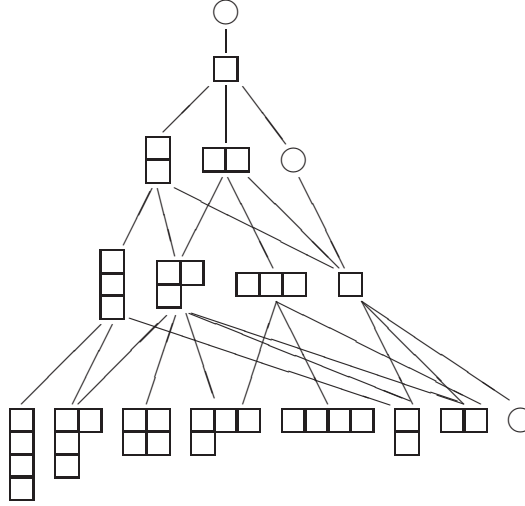


Figure 2: The Bratteli diagram for BW_t up to the 4^{th} level

We say that R is a path of length t in the Bratteli diagram of BW_t if R is a sequence of $t+1$ Young diagrams: $R = ([0], [1], \dots, r_t)$ where $r_s \in \Gamma_s$ for each $s = 0, 1, \dots, t$ and where r_i is connected to r_{i+1} for each $0 \leq i \leq t-1$. We say that $shp(R) = r_t$. Let Ω_t be the set of pairs (R, S) of paths of length t in the Bratteli diagram of BW_t where $r_t = s_t$, ie those paths R and S of length t where $shp(R) = shp(S)$.

Ram and Wenzl [21] wrote down a basis $\{e_{ST} \in BW_t \mid (S, T) \in \Omega_t\}$ of BW_t which is also a set of matrix units. We recall this set of matrix units below.

Let us fix some notation. Given a sequence $T = (0, s_1, \dots, s_t)$, we fix $T' = (0, s_1, \dots, s_{t-1})$. If T is a path of length t , then T' is the path of length $t-1$ obtained by removing the last vertex and edge of T .

Before defining the matrix units of BW_t we define some ‘pre-matrix units’ that we will employ in defining the matrix units. Let T be a path of length t in the Bratteli diagram for BW_t such that $shp(T)$ has t boxes. We can identify T with a standard tableau containing the numbers $1, 2, \dots, t$ in a canonical way. We do this by placing the number 1 in the top left hand box of $shp(T)$ and we then fill each box of $shp(T)$ with increasing numbers according to the path T [22, Sec. 4.2].

For each path T of length t in the Bratteli diagram for BW_t , we define the number $d(T, i)$ for each $i = 1, 2, \dots, t-1$ by

$$d(T, i) = c(i+1) - c(i) - r(i+1) + r(i), \quad (45)$$

where $c(j)$ and $r(j)$ denote the column and row, respectively, of the box containing the number j in the standard tableau corresponding to T . For each $d \in \mathbb{Z} \setminus \{0\}$, we define

$$b_d(q) = \frac{q^d(1-q)}{1-q^d}.$$

Let T be a path of length t in the Bratteli diagram for BW_t . Firstly fix the pre-matrix unit $o_{[1]} = 1 \in BW_t$. Let R be a path of length $t-1$ defined by $R = T'$ and inductively define

$$o_T = \prod_S \frac{o_R g_{t-1} o_R - b_{d(S,t-1)}(q^2) o_R}{b_{d(T,t-1)}(q^2) - b_{d(S,t-1)}(q^2)} \in BW_t,$$

where the product is over all paths S of length t where $\text{shp}(S)$ contains t boxes such that $S \neq T$ and $S' = R$. We write $o_{TT} = o_T$.

If M and P are paths of length t in the Bratteli diagram for BW_t where $(M, P) \in \Omega_t$ and $\text{shp}(M) = \text{shp}(P)$ has exactly t boxes and $\text{shp}(M') = \text{shp}(P')$, then we define

$$o_{MP} = o_{M'P'} o_{PP}.$$

Now, if M and P are paths of length t where $(M, P) \in \Omega_t$ and $\text{shp}(M) = \text{shp}(P)$ has exactly t boxes and $\text{shp}(M') \neq \text{shp}(P')$, then the pre-matrix unit o_{MP} is defined more intricately. To define o_{MP} , choose paths \overline{M} and \overline{P} of length t that satisfy $\text{shp}(\overline{M}) = \text{shp}(M)$, $\text{shp}(\overline{P}) = \text{shp}(P)$ and the following three conditions:

- (i) $\overline{M}'' = \overline{P}''$,
- (ii) $\text{shp}(\overline{M}') = \text{shp}(M')$,
- (iii) $\text{shp}(\overline{P}') = \text{shp}(P')$.

It may appear that these conditions cannot always be satisfied. However, paths \overline{M} and \overline{P} satisfying these conditions can always be constructed as follows [21].

By considering \overline{M} and \overline{P} as standard tableaux, we obtain the desired paths \overline{M} and \overline{P} by ensuring the following is true. Firstly, fix t to be in the same box in \overline{M} (resp. \overline{P}) that t is in M (resp. P). Then, fix $(t-1)$ to be in the same box in \overline{M} (resp. \overline{P}) that t is in P (resp. M). Lastly, for each $i = 1, 2, \dots, t-2$, fix i to be in the same box in \overline{M} that it is in \overline{P} .

We then define

$$o_{MP} = \frac{1 - q^{2d}}{\sqrt{(1 - q^{2(d+1)})(1 - q^{2(d-1)})}} o_{M'\overline{M}'} g_{t-1} o_{\overline{P}'P'} o_{PP},$$

where $d = d(\overline{M}, t-1)$ is as given in (45).

This completes the definition of the ‘pre-matrix units’; now we define the matrix units proper for BW_t .

Assume that the matrix units are known for BW_{t-1} . Let M and P be paths of length t in the Bratteli diagram for BW_t where $\text{shp}(M) = \lambda = \text{shp}(P)$ and λ contains strictly fewer than t boxes, then we define

$$e_{MP} = \frac{Q_\lambda(r, q)}{\sqrt{Q_\mu(r, q) Q_{\tilde{\mu}}(r, q)}} e_{M'S} e_{t-1} e_{TP'},$$

where S and T are paths of length $t-1$ satisfying

- (i) $\text{shp}(S) = \text{shp}(M') = \mu$, and
- (ii) $\text{shp}(T) = \text{shp}(P') = \tilde{\mu}$, and
- (iii) $S' = T'$, and
- (iv) $\text{shp}(S') = \lambda = \text{shp}(T')$.

It may appear that these conditions cannot always be satisfied. However, there always exists a pair of paths S and T of length $t-1$ satisfying these conditions for the following reasons. Firstly, by examining the relevant Bratteli diagrams, it is clear that there are no intertwiner matrix units in BW_1 and BW_2 . Now for each $t \geq 3$, a shape λ that has at most $t-2$ boxes and which labels a

vertex on the t^{th} level of the Bratteli diagram for BW_t also labels a vertex on the $(t-2)^{nd}$ level of the Bratteli diagram. Hence there always exists at least one path of length $t-2$ in the Bratteli diagram for BW_t ending at the vertex $shp(M)$ on the $(t-2)^{nd}$ level, as $shp(M)$ contains no more than $t-2$ boxes (note that this shows that (iii) and (iv) might be satisfied).

In the Bratteli diagram for BW_t , two vertices λ and μ are connected by an edge only if their shapes differ by exactly one box. Now the vertices $shp(M')$ and $shp(P')$ on the $(t-1)^{st}$ level are connected to the vertex $shp(M)$ on the t^{th} level by one edge each, and they are also connected to the vertex $shp(M)$ on the $(t-2)^{nd}$ level by one edge each. It follows, then, that by fixing S and T to be paths of length $t-1$ that coincide on the first $t-2$ levels of the Bratteli diagram and that pass through the vertex $shp(M)$ on the $(t-2)^{nd}$ level, and also fixing $shp(S) = shp(M')$ and $shp(T) = shp(P')$ (which is always possible), we obtain the desired paths S and T .

Let M and P be paths of length t in the Bratteli diagram for BW_t , where $(M, P) \in \Omega_t$, and where $shp(M)$ contains t boxes. Then we define

$$e_{MP} = (1 - z_t) o_{MP},$$

where $z_t = \sum_P e_{PP}$ with the summation going over all paths P of length t such that $shp(P)$ contains fewer than t boxes.

The following fact is important [24, Lem. 4.2]: let M be a path of length t in the Bratteli diagram for BW_t where $shp(M) = \lambda$, then

$$\text{tr}(e_{MM}) = Q_\lambda(r, q)/x^t, \quad (46)$$

where $x = \frac{r - r^{-1}}{q - q^{-1}} + 1$ and $Q_\lambda(r, q)$ is the function given below in (48).

It is interesting to note that the quantum superdimension of the fundamental irreducible $U_q(\mathfrak{osp}(1|2n))$ -module V is $(-q^{2n} + q^{-2n})/(q - q^{-1}) + 1$, which is just the expression x with the indeterminates r and q replaced with the complex numbers $-q^{2n}$ and q , respectively. Recall that we grade the highest weight vector of V to be odd.

11.3 The algebra $BW_t(r, q)$

The algebra $BW_t(r, q)$, with $r, q \in \mathbb{C}$, is equipped with a functional $\text{tr} : BW_t(r, q) \rightarrow \mathbb{C}$ which satisfies, amongst other relations,

$$\text{tr}(a\chi b) = \text{tr}(\chi)\text{tr}(ab), \quad \forall a, b \in BW_{t-1}(r, q), \quad \chi \in \{g_{t-1}, e_{t-1}\}, \quad (47)$$

where we regard each element of $BW_{t-1}(r, q)$ as an element of $BW_t(r, q)$ under the canonical inclusion.

Define the annihilator ideal $J_t(r, q) \subset BW_t(r, q)$ with respect to tr by

$$J_t(r, q) = \{b \in BW_t(r, q) \mid \text{tr}(ab) = 0, \forall a \in BW_t(r, q)\}.$$

If q is not a root of unity and $r \neq \pm q^k$ for all integers k , then $J_t(r, q) = 0$ and $BW_t(r, q)$ is semisimple [24, Cor. 5.6]. If $r = \pm q^k$ for some $k \in \mathbb{Z}$, then $J_t(\pm q^k, q) \neq 0$ and the quotient $BW_t(\pm q^k, q)/J_t(\pm q^k, q)$ is semisimple [24, Cor. 5.6].

Let us now fix $k = 2n$ and $r = -q^{2n}$; recall that the homomorphism $\Upsilon : g_i \mapsto -\check{R}_i \in \mathcal{C}_t$ yields a representation of $BW_t(-q^{2n}, q)$ in \mathcal{C}_t . The next task is to determine the structure of $BW_t(-q^{2n}, q)/J_t(-q^{2n}, q)$, which we do in the following work from [24].

Let us now introduce a subgraph $\Gamma(-q^{2n}, q)$ of the Young lattice that we will use in describing the structure of $BW_t(-q^{2n}, q)/J_t(-q^{2n}, q)$. We inductively obtain the vertices of $\Gamma(-q^{2n}, q)$ as follows. Firstly fix the Young diagram with no boxes to belong to $\Gamma(-q^{2n}, q)$. The inductive step is that if the Young diagram μ belongs to $\Gamma(-q^{2n}, q)$, the Young diagram λ also belongs to $\Gamma(-q^{2n}, q)$ if λ differs from μ by exactly one box and if $Q_\lambda(-q^{2n}, q) \neq 0$, where $Q_\lambda(r, q)$ is given in the next paragraph.

Given a Young diagram λ , let (i, j) denote the box in the i^{th} row and the j^{th} column of λ , and let λ_i (resp. λ'_j) denote the number of boxes in the i^{th} row (resp. j^{th} column) of λ . We introduce some useful notation: we may denote the Young diagram λ by $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$ where the i^{th} row contains λ_i boxes for each $i = 1, 2, \dots, k$, and the l^{th} row contains no boxes for each $l > k$. The function $Q_\lambda(r, q)$ is

$$Q_\lambda(r, q) = \prod_{(j, j) \in \lambda} \frac{rq^{\lambda_j - \lambda'_j} - r^{-1}q^{-\lambda_j + \lambda'_j} + q^{\lambda_j + \lambda'_j - 2j + 1} - q^{-\lambda_j - \lambda'_j + 2j - 1}}{q^{h(j, j)} - q^{-h(j, j)}} \times \prod_{(i, j) \in \lambda, i \neq j} \frac{rq^{d(i, j)} - r^{-1}q^{-d(i, j)}}{q^{h(i, j)} - q^{-h(i, j)}}, \quad (48)$$

where the hooklength $h(i, j)$ is defined by $h(i, j) = \lambda_i - i + \lambda'_j - j + 1$, and where

$$d(i, j) = \begin{cases} \lambda_i + \lambda_j - i - j + 1, & \text{if } i \leq j, \\ -\lambda'_i - \lambda'_j + i + j - 1, & \text{if } i > j. \end{cases}$$

Intuitively, the hooklength $h(i, j)$ is the number of boxes in the hook going through the box (i, j) , ie the number of boxes below the (i, j) box in the j^{th} column plus the number of boxes to the right of the (i, j) box in the i^{th} row, plus one.

Now $h(i, j) \geq 1$ for all $(i, j) \in \lambda$, so $Q_\lambda(-q^{2n}, q)$ is well-defined for all λ . Also, for each $(j, j) \in \lambda$ we have

$$\begin{aligned} & -q^{2n + \lambda_j - \lambda'_j} + q^{-2n - \lambda_j + \lambda'_j} + q^{\lambda_j + \lambda'_j - 2j + 1} - q^{-\lambda_j - \lambda'_j + 2j - 1} \\ &= (q^{-n + \lambda'_j - j + 1/2} - q^{n - \lambda'_j + j - 1/2})(q^{n + \lambda_j - j + 1/2} + q^{-n - \lambda_j + j - 1/2}), \end{aligned}$$

and so $Q_\lambda(-q^{2n}, q) = 0$ if and only if at least one of the following conditions is satisfied:

- (a) $q^{4n + 2d(i, j)} = 1$ for some $(i, j) \in \lambda$ where $i \neq j$,
- (b) $q^{2n - 2\lambda'_j + 2j - 1} = 1$ or $q^{2n + 2\lambda_j - 2j + 1} = -1$ for some j .

Now $Q_\lambda(-q^{2n}, q) = Q_\lambda((-q)^{2n}, -q)$ from Lemma 11.5, thus

$\Gamma(-q^{2n}, q) = \Gamma((-q)^{2n}, -q)$. Wenzl completely determined $\Gamma((-q)^{2n}, -q)$ in [24, Cor. 5.6 (c), (c1)], which we restate here with a correction. Firstly, [24, Cor. 5.6 (c)] reads

(c) ‘If $r = q^n$ and q is not a root of unity, $\Gamma(r, q)$ consists of all Young diagrams λ for which’

and [24, Cor. 5.6 (c1)] is presented with a slight error; it should read

(c1) ‘ $\lambda'_1 + \lambda'_2 \leq n + 1$ for $n > 0$.’

Then both $\Gamma((-q)^{2n}, -q)$ and $\Gamma(-q^{2n}, q)$ consist of all Young diagrams λ satisfying $\lambda'_1 + \lambda'_2 \leq 2n + 1$. Let us determine $\Gamma(-q^{2n}, q)$ independently of [24, Cor. 5.6 (c), (c1)].

Now q is non-zero and not a root of unity, so (b) above is never satisfied for any λ , and (a) is only satisfied if $d(i, j) = -2n$. We now determine the circumstances in which $d(i, j) = -2n$. If $i > j$, we can see that $\min(d(i, j)) = d(2, 1) = -\lambda'_1 - \lambda'_2 + 2$ from the constraints on the lengths of the columns of a Young diagram and it follows that $Q_\lambda(-q^{2n}, q) = 0$ if $\lambda'_1 + \lambda'_2 = 2n + 2$. Let us call a Young diagram λ *allowable* if $\lambda'_1 + \lambda'_2 \leq 2n + 1$.

Across all the allowable Young diagrams, let us calculate $\min(d(i, j))$ where $i < j$. If the first column of the allowable diagram λ contains $2n + 1$ boxes, ie $\lambda'_1 = 2n + 1$, then all the other columns must contain no boxes from the definition of an allowable diagram. For such a λ , there does not exist any box (i, j) in the i^{th} row and the j^{th} column with $i < j$ and so there is nothing more to consider in this case. Now if the first column of λ contains strictly fewer than $2n + 1$ boxes, ie $\lambda'_1 \leq 2n$, then the following relations hold: $i \leq 2n$, $\lambda_i - j \geq 0$ and $\lambda_j \geq 0$. Then $d(i, j) = \lambda_i + \lambda_j - i - j + 1 \geq -2n + 1$, which means that $d(i, j) \neq -2n$ for all $i < j$.

It follows that $Q_\lambda(-q^{2n}, q) = 0$ if $\lambda'_1 + \lambda'_2 = 2n + 2$ and that $Q_\lambda(-q^{2n}, q) \neq 0$ for all allowable Young diagrams λ . Consequently, the vertices of $\Gamma(-q^{2n}, q)$ are all the allowable Young diagrams, that is, all the Young diagrams λ satisfying $\lambda'_1 + \lambda'_2 \leq 2n + 1$, and thus $\Gamma(-q^{2n}, q)$ is indeed identical to $\Gamma((-q)^{2n}, -q)$. In Figure 11.3 we show the graph $\Gamma(-q^{2n}, q)$ with $n = 1$ up to the 4th level.

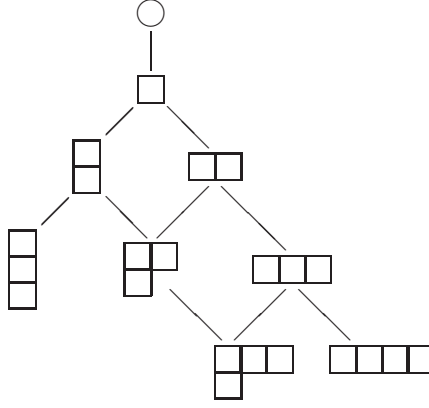


Figure 3: The graph $\Gamma(-q^{2n}, q)$ with $n = 1$ up to the 4th level

Now $J_t(-q^{2n}, q) \neq 0$ and $BW_t(-q^{2n}, q)$ is not semisimple. However, the quotient $BW_t(-q^{2n}, q)/J_t(-q^{2n}, q)$ is semisimple:

$$BW_t(-q^{2n}, q)/J_t(-q^{2n}, q) \cong \bigoplus_{\lambda \in \Gamma_t(-q^{2n}, q)} M_{b_\lambda}(\mathbb{C}),$$

where $\Gamma_t(-q^{2n}, q)$ is the set of Young diagrams belonging to $\Gamma(-q^{2n}, q)$ with $t - 2k \geq 0$ boxes, where k ranges over all of \mathbb{Z}_+ [24, Cor. 5.6].

We obtain the matrix units in $BW_t(-q^{2n}, q)/J_t(-q^{2n}, q)$ by taking a certain proper subset of the matrix units in BW_t and replacing the indeterminates r and q with the complex numbers $-q^{2n}$ and q , respectively.

To label the matrix units for $BW_t(-q^{2n}, q)/J_t(-q^{2n}, q)$, we use the Bratteli diagram for $BW_t(-q^{2n}, q)/J_t(-q^{2n}, q)$, which we define in the same way as we defined the Bratteli diagram for BW_t but we replace each Γ_s with the set $\Gamma_s(-q^{2n}, q)$ as follows. Recall that $\Gamma(-q^{2n}, q)$ is a subgraph of the Young lattice the vertices of which are all the allowable Young diagrams. Then, for each $s = 0, 1, \dots, t$, we fix $\Gamma_s(-q^{2n}, q)$ to be the set of all Young diagrams that are vertices of $\Gamma(-q^{2n}, q)$ that contain exactly $s - 2k \geq 0$ boxes, where k ranges over all of \mathbb{Z}_+ . In Figure 11.3 we show the Bratteli diagram for $BW_t(-q^{2n}, q)/J_t(-q^{2n}, q)$ with $n = 1$ up to the 4th level.

We say that $T = (0, s_1, \dots, s_t)$ is a path of length t in the Bratteli diagram for $BW_t(-q^{2n}, q)/J_t(-q^{2n}, q)$ if $s_i \in \Gamma_i(-q^{2n}, q)$ for each i and if s_j is joined by an edge to s_{j+1} for each $j = 0, 1, \dots, t-1$. Note that s_j is joined to s_{j+1} by an edge if and only if s_j differs from s_{j+1} by exactly one box.

Let $\Omega_t(-q^{2n}, q)$ be the set of pairs (R, S) of paths of length t in the Bratteli diagram for $BW_t(-q^{2n}, q)/J_t(-q^{2n}, q)$ where $r_t = s_t$, that is $shp(R) = shp(S)$. The matrix units

$$\{e_{RS} \in BW_t \mid (R, S) \in \Omega_t(-q^{2n}, q)\}$$

are all well-defined and non-zero upon the indeterminates r and q being replaced with the complex numbers $-q^{2n}$ and q , respectively. Henceforth we write e_{RS} to mean the matrix unit $e_{RS}(-q^{2n}, q) \in BW_t(-q^{2n}, q)/J_t(-q^{2n}, q)$. It is very important to note that $\text{tr}(e_{SS}) \neq 0$ for all $(S, S) \in \Omega_t(-q^{2n}, q)$ and that $e_{RS} \notin J_t(-q^{2n}, q)$ for all $(R, S) \in \Omega_t(-q^{2n}, q)$.

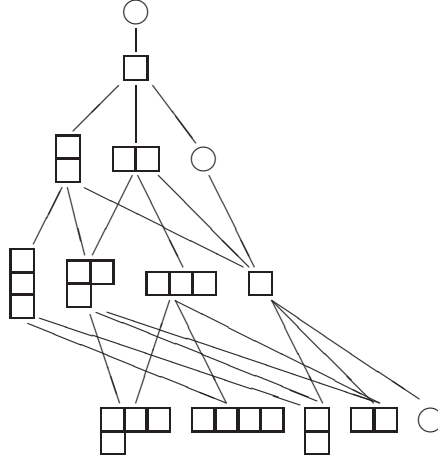


Figure 4: The Bratteli diagram for $BW_t(-q^{2n}, q)/J_t(-q^{2n}, q)$ with $n = 1$ up to the 4th level

11.4 Matrix units in $BW_t(-q^{2n}, q)/J_t(-q^{2n}, q)$ and \mathcal{C}_t

We now relate the idempotent matrix units in $BW_t(-q^{2n}, q)/J_t(-q^{2n}, q)$ to the projectors in \mathcal{C}_t we defined at the start of this section. Let $\widetilde{BW}_t(-q^{2n}, q)$ be the semisimple subalgebra of $BW_t(-q^{2n}, q)$ spanned by the matrix units in $BW_t(-q^{2n}, q)/J_t(-q^{2n}, q)$, ie $\{e_{RS} \mid (R, S) \in \Omega_t(-q^{2n}, q)\}$.

Firstly, we will show that $BW_t(-q^{2n}, q) = \widetilde{BW}_t(-q^{2n}, q) \oplus J_t(-q^{2n}, q)$. Any $f \in \widetilde{BW}_t(-q^{2n}, q)$ can be written as

$$f = \sum_{(S,T) \in \Omega_t(-q^{2n}, q)} f_{ST} e_{ST}, \quad f_{ST} \in \mathbb{C},$$

where $f_{ST} \neq 0$ for at least one pair (S, T) of paths. Fix (A, B) to be such a pair, then

$$\text{tr}(e_{BA}f) = \text{tr}(f_{AB}e_{BA}e_{AB}) = f_{AB}\text{tr}(e_{BB}) \neq 0,$$

as $\text{tr}(e_{BB}) \neq 0$. Thus any non-zero f belonging to $\widetilde{BW}_t(-q^{2n}, q)$ does not belong to $J_t(-q^{2n}, q)$, yielding

$$BW_t(-q^{2n}, q) = \widetilde{BW}_t(-q^{2n}, q) \oplus J_t(-q^{2n}, q).$$

Then we can write $a = \tilde{a} + a_j$ for each $a \in BW_t(-q^{2n}, q)$, where $\tilde{a} \in \widetilde{BW}_t(-q^{2n}, q)$ and $a_j \in J_t(-q^{2n}, q)$.

Now define

$$P_t = \sum_{(S,S) \in \Omega_t(-q^{2n}, q)} e_{SS} \in \widetilde{BW}_t(-q^{2n}, q),$$

then $P_t a P_t = \tilde{a}$, which can be seen by regarding $BW_t(-q^{2n}, q)$ as a matrix algebra.

Let us now turn our attention to \mathcal{C}_t . Define $\mathcal{J}_t \subset \mathcal{C}_t$ to be the annihilator ideal of \mathcal{C}_t with respect to the quantum supertrace:

$$\mathcal{J}_t = \{b \in \mathcal{C}_t \mid \text{str}_q(ab) = 0, \forall a \in \mathcal{C}_t\}.$$

Now define a map $\psi : \mathcal{C}_t \rightarrow \mathbb{C}$ by

$$\psi(X) = \text{str}_q(X) / (\text{sdim}_q(V))^t,$$

then $\psi(X) = 0$ if and only if $\text{str}_q(X) = 0$, and furthermore,

$$\psi(\Upsilon(a)) = \text{tr}(a), \quad \forall a \in BW_t(-q^{2n}, q), \quad (49)$$

from Lemma 11.7. Thus we can regard \mathcal{J}_t as the annihilator ideal of \mathcal{C}_t with respect to ψ .

Remark 11.2. *The equality of the traces in (49) independently confirms that link invariants can be created from representations of $U_q(\mathfrak{osp}(1|2n))$ as Zhang did in [26].*

Now we will use Eq. (49) to show that

$$\Upsilon(J_t(-q^{2n}, q)) = \mathcal{J}_t. \quad (50)$$

We firstly show that $\Upsilon(J_t(-q^{2n}, q)) \subseteq \mathcal{J}_t$. Let b be an arbitrary element of $J_t(-q^{2n}, q)$, then $\text{tr}(ab) = 0$ for all $a \in BW_t(-q^{2n}, q)$, and the surjectivity of Υ , in addition to the fact that $\psi(\Upsilon(ab)) = \text{tr}(ab)$, means that $\Upsilon(b) \in \mathcal{J}_t$.

Now let B be an arbitrary element of \mathcal{J}_t , then there exists some $b \in BW_t(-q^{2n}, q)$ satisfying $B = \Upsilon(b)$. Furthermore, $b \in J_t(-q^{2n}, q)$ as $\text{tr}(ab) = \psi(\Upsilon(a)\Upsilon(b)) = 0$ for all $a \in BW_t(-q^{2n}, q)$. Then $\Upsilon(J_t(-q^{2n}, q)) \supseteq \mathcal{J}_t$, proving (50).

The surjectivity of Υ implies that

$$\mathcal{C}_t = \Upsilon(\widetilde{BW}_t(-q^{2n}, q)) + \mathcal{J}_t,$$

and we will show that this sum is in fact direct. To see this, assume that there exists some non-zero element F of \mathcal{C}_t belonging to $\Upsilon(\widetilde{BW}_t(-q^{2n}, q))$ and also belonging to \mathcal{J}_t , then $\text{str}_q(XF) = 0$ for all $X \in \mathcal{C}_t$. However, F is the image of a linear combination of matrix units:

$$F = \sum_{(S,T) \in \Omega_t(-q^{2n}, q)} f_{ST} \Upsilon(e_{ST}), \quad f_{ST} \in \mathbb{C},$$

where $f_{ST} \neq 0$ for at least one pair (S, T) . Assume that (A, B) is such a pair, then by similar reasoning as previously, $\text{str}_q(\Upsilon(e_{BA})F) \neq 0$ which contradicts the assumption that $F \in \mathcal{J}_t$. Thus $\Upsilon(\widetilde{BW}_t(-q^{2n}, q)) \cap \mathcal{J}_t = 0$, and

$$\mathcal{C}_t = \Upsilon(\widetilde{BW}_t(-q^{2n}, q)) \oplus \mathcal{J}_t. \quad (51)$$

It is clear that the image of the collection of all matrix units $e_{ST} \in \widetilde{BW}_t(-q^{2n}, q)$ in \mathcal{C}_t under the map Υ is again a collection of matrix units. Each matrix unit $e_{SS} \in \widetilde{BW}_t(-q^{2n}, q)$ is an idempotent, thus each $\Upsilon(e_{SS})$ is an idempotent that is also a $U_q(\mathfrak{g})$ -linear map. The idempotents $\{\Upsilon(e_{SS}) \mid (S, S) \in \Omega_t(-q^{2n}, q)\}$ are all orthogonal as the matrix units $\{e_{SS} \mid (S, S) \in \Omega_t(-q^{2n}, q)\}$ are all orthogonal.

Let S be a path of length t in the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$ and let T be a path of length t in the Bratteli diagram for $V^{\otimes t}$. Both $\Upsilon(e_{SS})$ and the projection $E_{TT} \in \mathcal{C}_t$ project down onto the same decomposition of $V^{\otimes t}$. To see this, let us write $S = (0, s_1, \dots, s_t)$ and $T = (0, t_1, \dots, t_t)$ and fix $S_j = (0, s_1, \dots, s_j)$ and $T_j = (0, t_1, \dots, t_j)$ for each $j = 1, 2, \dots, t$. For each j , $\Upsilon(e_{S_j S_j}) \in \mathcal{C}_j$ and $E_{T_j T_j} \in \mathcal{C}_j$ are embedded into \mathcal{C}_t via $Z \mapsto Z \otimes \text{id}^{\otimes(t-j)}$, and they are recursively defined so that $\Upsilon(e_{S_{j-1} S_{j-1}})$ and $E_{T_{j-1} T_{j-1}}$ explicitly appear in the definitions of $\Upsilon(e_{S_j S_j})$ and $E_{T_j T_j}$, respectively. Lastly, note that both $\Upsilon(e_{S_j S_j})$ and $E_{T_j T_j}$ project down onto the j left-most tensor powers of V in $V^{\otimes t}$.

Fix $e_{SS} \in \widetilde{BW}_t(-q^{2n}, q)$ to be an idempotent matrix unit where $(S, S) \in \Omega_t(-q^{2n}, q)$. Now $\Upsilon(e_{SS})(V^{\otimes t}) \neq 0$ as $\text{str}_q(\Upsilon(e_{SS})) = (\text{sdim}_q(V))^t \text{tr}(e_{SS}) \neq 0$, and as $V^{\otimes t}$ is completely reducible, $\Upsilon(e_{SS})$ projects down from $V^{\otimes t}$ onto a direct sum of irreducible $U_q(\mathfrak{g})$ -submodules $W_{\text{shp}(S)}$ of $V^{\otimes t}$.

Let us write $\lambda = \text{shp}(S)$ and define the Young diagram λ^* corresponding to λ by

$$\lambda^* = \begin{cases} \lambda, & \text{if } \lambda'_1 \leq n, \\ \tilde{\lambda}, & \text{if } \lambda'_1 \geq n+1, \end{cases} \quad (52)$$

where $\tilde{\lambda}$ is given in Lemma 11.4. We will inductively show that:

Lemma 11.1. $W_{shp(S)}$ is an irreducible $U_q(\mathfrak{g})$ -submodule of $V^{\otimes t}$ with integral dominant highest weight λ^* , where the Young diagram λ^* is taken to be a highest weight as discussed in the third paragraph of Subsection 11.1.

Proof. Firstly, let $R = (0, \epsilon_1)$ be a path of length 1; $e_{RR} = 1$ and $\Upsilon(e_{RR})$ acts as the identity on V , the fundamental irreducible $U_q(\mathfrak{g})$ -module with integral dominant highest weight ϵ_1 .

We now do the inductive step. Assume that R is a path of length $t-1$ where $(R, R) \in \Omega_{t-1}(-q^{2n}, q)$ and such that the idempotent $\Upsilon(e_{RR}) \in \mathcal{C}_{t-1}$ projects down from $V^{\otimes(t-1)}$ onto an irreducible $U_q(\mathfrak{g})$ -submodule $V_{\mu^*} \subseteq V^{\otimes(t-1)}$ with integral dominant highest weight μ^* , where μ^* is the weight corresponding to $\mu = shp(R)$ as given in (52). Let S be a path of length t where $(S, S) \in \Omega_t(-q^{2n}, q)$ and such that $S' = R$ and let $\lambda = shp(S)$, then $\Upsilon(e_{SS}) \in \mathcal{C}_t$ projects down from $V^{\otimes t}$ onto exactly one irreducible $U_q(\mathfrak{g})$ -submodule W_λ of $V_{\mu^*} \otimes V$ from Lemma 11.2. We now show that the highest weight of W_λ is the integral dominant weight λ^* .

From (46) and (49), $sdim_q(W_\lambda) = Q_\lambda(-q^{2n}, q)$, and from Lemma 11.5, $Q_\lambda(-q^{2n}, q) = Q_\lambda(q^{2n}, -q)$. If $\lambda'_1 \geq n+1$, then $Q_\lambda(q^{2n}, -q) = Q_\lambda(q^{2n}, -q)$ from [24, p. 422], and it follows that $Q_{\lambda^*}(q^{2n}, -q) = Q_\lambda(q^{2n}, -q)$.

It is well known that $Q_{\lambda^*}(q^{2n}, -q)$ is the quantum dimension of a finite dimensional irreducible $U_{-q}(so(2n+1))$ -module $V_{\lambda^*}^{so(2n+1)}$ with highest weight λ^* :

$$dim_{-q} \left(V_{\lambda^*}^{so(2n+1)} \right) = Q_{\lambda^*}(q^{2n}, -q).$$

From [27], there exists a finite dimensional irreducible $U_q(osp(1|2n))$ -module V_{λ^*} with highest weight λ^* with the same weight space decomposition as $V_{\lambda^*}^{so(2n+1)}$. The quantum superdimension of V_{λ^*} from [27, Eq. (15)] satisfies

$$sdim_q(V_{\lambda^*}) = dim_{-q} \left(V_{\lambda^*}^{so(2n+1)} \right),$$

and thus

$$sdim_q(W_\lambda) = Q_{\lambda^*}(-q^{2n}, q) = Q_{\lambda^*}(q^{2n}, -q) = sdim_q(V_{\lambda^*}). \quad (53)$$

Note that we have not yet proved that the highest weight of the irreducible $U_q(\mathfrak{g})$ -module W_λ is λ^* . Recall that $V_{\mu^*} \otimes V$ is completely reducible; from Lemma 11.3, the quantum superdimensions of the irreducible $U_q(\mathfrak{g})$ -submodules of $V_{\mu^*} \otimes V$ are all different, thus the highest weight of W_λ is indeed λ^* from (53). This completes the proof. \square

If R and S are paths of length t in the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$ satisfying $shp(R) = \lambda$ and $shp(S) = \tilde{\lambda}$, where $\tilde{\lambda}$ is given in Lemma 11.4, then $\Upsilon(e_{RR})$ and $\Upsilon(e_{SS})$ project down from $V^{\otimes t}$ onto isomorphic irreducible $U_q(\mathfrak{g})$ -submodules of $V^{\otimes t}$. However, we show in the next paragraph that no such paths exist in the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$ using an easy even/odd counting argument.

If the number of boxes in λ is even (resp. odd), then the number of boxes in $\tilde{\lambda}$ is odd (resp. even), as

$$\tilde{\lambda}'_1 \bmod 2 = (2n+1 - \lambda'_1) \bmod 2 = (1 - \lambda'_1) \bmod 2, \quad \text{and} \quad \tilde{\lambda}'_j = \lambda'_j, \quad j \geq 2.$$

Now let r be an even (resp. odd) number satisfying $0 \leq r \leq t$, then the vertices on the r^{th} level of the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$ are all the Young diagrams in $\Gamma(-q^{2n}, q)$ with $k \geq 0$ boxes where $k \leq r$ is an even (resp. odd) number. If $|\lambda|$ denotes the number of boxes in the Young diagram λ , then the fact that $|\lambda| \bmod 2 = (|\tilde{\lambda}| + 1) \bmod 2$ means that λ and $\tilde{\lambda}$ cannot both be vertices on the same level of the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$, and it follows that at most only one of the paths R and S can exist.

Let R be a path of length t in the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$. Let S be the same path of length t as R except that we replace each Young diagram λ on the path R with more than n rows of boxes with the Young diagram $\tilde{\lambda}$ given in (54). Then S is a path of length t in the Bratteli

diagram for $V^{\otimes t}$, and we have $E_{SS} = \Upsilon(e_{RR})$, ie the projectors E_{SS} and $\Upsilon(e_{RR})$ project from $V^{\otimes t}$ down onto the same $U_q(\mathfrak{g})$ -submodule.

We now present some technical lemmas we used in this work.

Lemma 11.2. *Let R be a path of length $t-1$ where $(R, R) \in \Omega_{t-1}(-q^{2n}, q)$ such that the idempotent $\Upsilon(e_{RR})$ projects down from $V^{\otimes(t-1)}$ onto an irreducible $U_q(\mathfrak{g})$ -submodule $V_\mu \subseteq V^{\otimes(t-1)}$ with integral dominant highest weight μ . Let S be a path of length t where $(S, S) \in \Omega_t(-q^{2n}, q)$ such that $S' = R$, then $\Upsilon(e_{SS})$ projects down from $V^{\otimes t}$ onto exactly one irreducible $U_q(\mathfrak{g})$ -submodule of $V_\mu \otimes V$.*

Proof. Recall that $V_\mu \otimes V$ is completely reducible. Assume that $\Upsilon(e_{SS})$ projects down onto at least two irreducible $U_q(\mathfrak{g})$ -submodules of $V_\mu \otimes V$:

$$\Upsilon(e_{SS})(V_\mu \otimes V) = \bigoplus_{\nu \in I} V_\nu,$$

for some index set I . Then, by construction, there exists a projection $E_{MM_\nu} \in \mathcal{C}_t$ for each $\nu \in I$ such that $E_{MM_\nu}(V_\mu \otimes V) = V_\nu \subseteq V_\mu \otimes V$.

From (51), each E_{MM_ν} can be written as a sum $E_{MM_\nu} = \tilde{E}_{MM_\nu} + E_{MM_\nu}^j$ where

$$\tilde{E}_{MM_\nu} = \sum_{(S,T) \in \Omega_t(-q^{2n}, q)} c_{ST} \Upsilon(e_{ST}) \in \Upsilon(\widetilde{BW}_t(-q^{2n}, q)),$$

and $E_{MM_\nu}^j \in \Upsilon(J_t(-q^{2n}, q))$. Note that $\tilde{E}_{MM_\nu} \neq 0$ as $\text{str}_q(E_{MM_\nu}) \neq 0$ and $\text{str}_q(E_{MM_\nu}^j) = 0$. From Lemma 11.6, for each projection $E_{SS} \in \mathcal{C}_t$ there exists at least one idempotent $e_{RR} \in \widetilde{BW}_t(-q^{2n}, q)$ such that

$$(E_{SS}V^{\otimes t}) \cap (\Upsilon(e_{RR})V^{\otimes t}) \neq 0.$$

Let $E_{SS}(V_\mu \otimes V) = V_\nu \subseteq V^{\otimes t}$ and fix e_{RR} to be such an idempotent, then

$$(E_{SS}V^{\otimes t}) \cap (\Upsilon(e_{RR})V^{\otimes t}) = V_\mu,$$

and $E_{SS} = \Upsilon(e_{RR}) + E_{SS}^j$ where $E_{SS}^j \in \Upsilon(J_t(-q^{2n}, q))$.

Now assume that there exists some non-zero projection $E_{TT} \in \mathcal{C}_t$ orthogonal to E_{SS} and satisfying

$$(E_{TT}V^{\otimes t}) \cap (\Upsilon(e_{RR})V^{\otimes t}) \neq 0,$$

then $E_{TT} = \Upsilon(e_{RR}) + E_{TT}^j$ where $E_{TT}^j \in \Upsilon(J_t(-q^{2n}, q))$. Now

$$E_{TT}E_{SS} = \Upsilon(e_{RR}) + \Upsilon(e_{RR})E_{SS}^j + E_{TT}^j\Upsilon(e_{RR}) + E_{TT}^jE_{SS}^j \neq 0,$$

which is non-vanishing as the first term in the sum is an element of $\Upsilon(\widetilde{BW}_t(-q^{2n}, q))$ and the last three terms in the sum are elements of $\Upsilon(J_t(-q^{2n}, q))$. However, the fact that $E_{TT}E_{SS} \neq 0$ contradicts the assumption that E_{TT} and E_{SS} are orthogonal, and thus such a non-zero E_{TT} cannot exist. However, such a non-zero E_{TT} must exist if $\Upsilon(e_{RR})$ projects onto the direct sum of at least two irreducible modules, thus $\Upsilon(e_{RR})$ projects only onto the zero vector or onto one irreducible $U_q(\mathfrak{g})$ -module. As $\text{str}_q(\Upsilon(e_{RR})) \neq 0$, $\Upsilon(e_{RR})$ projects onto one irreducible $U_q(\mathfrak{g})$ -module, completing the proof. \square

Lemma 11.3. *Let V_μ be a finite-dimensional irreducible $U_q(\mathfrak{g})$ -module with integral dominant highest weight $\mu \in \mathcal{P}^+$ and let*

$$V_\mu \otimes V \cong \bigoplus_{\lambda \in \mathcal{P}_\mu^+} V_\lambda$$

be the decomposition of $V_\mu \otimes V$ into irreducible $U_q(\mathfrak{g})$ -submodules V_λ . Then, for all $\lambda, \nu \in \mathcal{P}_\mu^+$, $\text{sdim}_q(V_\lambda) \neq \text{sdim}_q(V_\nu)$ if $\lambda \neq \nu$.

Proof. Recall that, by definition,

$$sdim_q(V_\lambda) = \text{str}(\pi_\lambda(K_{2\rho})) = \sum_{\phi} (-1)^{[w_\phi]} \text{mul}(\phi) q^{(2\rho, \phi)},$$

where the sum is over all weights ϕ of V_λ , $\text{mul}(\phi)$ is the multiplicity of the weight space ϕ in V_λ and $w_\phi \in V_\lambda$ is a weight vector of weight ϕ . By inspection, $(2\rho, \phi) \leq (2\rho, \lambda)$ for all weights ϕ of V_λ . To complete the proof, all we need show is that $(2\rho, \lambda) \neq (2\rho, \nu)$ for all integral dominant weights $\lambda, \nu \in \mathcal{P}_\mu^+$ where $\lambda \neq \nu$.

Recall that $\mathcal{P}_\mu^+ \subseteq \mathcal{P}_\mu^0 = \{\mu, \mu \pm \epsilon_i \in \mathcal{P}^+ \mid i = 1, 2, \dots, n\}$. It is then easy to show that $(2\rho, \beta) \neq (2\rho, \gamma)$ for all $\beta, \gamma \in \mathcal{P}_\mu^0$ where $\beta \neq \gamma$, which completes the proof. \square

Lemma 11.4. *Let λ be a Young diagram satisfying $\lambda'_1 \geq n+1$ and $\lambda'_1 + \lambda'_2 \leq 2n+1$. Define a diagram as a Young diagram is defined except that the length of the $(j+1)^{\text{st}}$ row (resp. column) of the diagram can be greater than the length of the j^{th} row (resp. column), for each j . Define the diagram $\tilde{\lambda}$ corresponding to λ by*

$$\tilde{\lambda}'_1 = 2n+1 - \lambda'_1, \quad \text{and} \quad \tilde{\lambda}'_j = \lambda'_j \quad \text{for all } j \geq 2. \quad (54)$$

Then $\tilde{\lambda}$ is a Young diagram.

Proof. Let λ be a Young diagram as given in the lemma and write $\lambda'_1 = \lambda'_2 + k$ where $k \geq 1$, then $\tilde{\lambda}'_1 = 2n+1 - (\lambda'_2 + k)$. Now $\tilde{\lambda}$ is a Young diagram if $\tilde{\lambda}'_1 \geq \tilde{\lambda}'_2$, and this condition is just $2n+1 - k \geq 2\lambda'_2$, which is true as $\lambda'_1 + \lambda'_2 = 2\lambda'_2 + k \leq 2n+1$. This completes the proof. \square

Lemma 11.5. *For each Young diagram λ ,*

$$Q_\lambda(-q^{2n}, q) = Q_\lambda(q^{2n}, -q). \quad (55)$$

Proof. Simple calculations show that (55) is true if and only if

$$\begin{aligned} \prod_{\substack{(i,j) \in \lambda \\ i \neq j}} -q^{2n+d(i,j)} + q^{-2n-d(i,j)} &= \left(\prod_{\substack{(i,j) \in \lambda \\ i < j}} (-1)^{\lambda'_j + \lambda_j} (q^{2n+d(i,j)} - q^{-2n-d(i,j)}) \right) \\ &\quad \times \left(\prod_{\substack{(i,j) \in \lambda \\ i > j}} (-1)^{\lambda_i + \lambda'_i} (q^{2n+d(i,j)} - q^{-2n-d(i,j)}) \right) \end{aligned}$$

and this last equation is true if

$$\prod_{\substack{(i,j) \in \lambda \\ i \neq j}} (-1) = \left(\prod_{\substack{(i,j) \in \lambda \\ i < j}} (-1)^{\lambda'_j + \lambda_j} \right) \left(\prod_{\substack{(i,j) \in \lambda \\ i > j}} (-1)^{\lambda_i + \lambda'_i} \right). \quad (56)$$

We now show that (56) is true. Define the following sets:

$$\begin{aligned} \text{Hor}_k &= \{(k, j) \in \lambda \mid j = 1, 2, \dots, \min\{k-1, \lambda_k\}\} \\ \text{Ver}_k &= \{(i, k) \in \lambda \mid i = 1, 2, \dots, \min\{k-1, \lambda'_k\}\}. \end{aligned}$$

Noting that $|\text{Ver}_k \cap \text{Hor}_l| = 0$ for all k and l and that $|\text{Hor}_k \cap \text{Hor}_l| = 0 = |\text{Ver}_k \cap \text{Ver}_l|$ for all $k \neq l$, (56) is true if the following equation is true for each k :

$$(-1)^{|\text{Hor}_k \cup \text{Ver}_k|} = \left(\prod_{(i,k) \in \text{Ver}_k} (-1)^{\lambda'_k + \lambda_k} \right) \left(\prod_{(k,j) \in \text{Hor}_k} (-1)^{\lambda_k + \lambda'_k} \right). \quad (57)$$

If $|\text{Hor}_k \cup \text{Ver}_k|$ is even, the right hand side of (57) clearly equals 1 as Hor_k and Ver_k are disjoint. Alternatively, if $|\text{Hor}_k \cup \text{Ver}_k|$ is odd, then $\lambda_k \leq k-2$ and/or $\lambda'_k \leq k-2$. If $\lambda'_k \leq k-2$, then $\lambda_k \leq k-1$ as λ is a Young diagram. Similarly, if $\lambda_k \leq k-2$, then $\lambda'_k \leq k-1$ as λ is a Young diagram. In both cases it follows that $\lambda'_k = |\text{Ver}_k|$ and $\lambda_k = |\text{Hor}_k|$. If $|\text{Hor}_k \cup \text{Ver}_k|$ is odd, $\lambda_k + \lambda'_k$ is also odd as $\lambda_k + \lambda'_k = |\text{Hor}_k| + |\text{Ver}_k| = |\text{Hor}_k \cup \text{Ver}_k|$, and clearly the right hand side of (57) equals -1 . Thus (57) is true for each k , from which it follows that (56) is true, which completes the proof of the lemma. \square

Lemma 11.6. *There does not exist a non-zero projector $E_{SS} \in \mathcal{C}_t$ that projects onto one irreducible $U_q(\mathfrak{g})$ -summand of $V^{\otimes t}$ with the following property:*

$$(E_{SS}V^{\otimes t}) \cap (\Upsilon(e_{RR})V^{\otimes t}) = 0, \quad (58)$$

for all idempotent matrix units $e_{RR} \in \widetilde{BW}_t(-q^{2n}, q)$.

Proof. Suppose that such a projector did exist, then $E_{SS}\Upsilon(e_{RR}) = 0$ for each idempotent matrix unit $e_{RR} \in \widetilde{BW}_t(-q^{2n}, q)$, and $E_{SS}\Upsilon(e_{RT}) = 0$ for each matrix unit $e_{RT} \in \widetilde{BW}_t(-q^{2n}, q)$ where $R \neq T$ and $(R, T) \in \Omega_t(-q^{2n}, q)$ as

$$E_{SS}\Upsilon(e_{RT}) = E_{SS}\Upsilon(e_{RR}e_{RT}e_{TT}) = 0 = \Upsilon(e_{RR}e_{RT}e_{TT})E_{SS} = \Upsilon(e_{RT})E_{SS},$$

as Υ is a homomorphism. We will show that it then follows that $E_{SS} \in \mathcal{J}_t$. However, the assumption that E_{SS} is a non-zero projector and that each irreducible $U_q(\mathfrak{g})$ -summand of $V^{\otimes t}$ has non-zero quantum superdimension means that $\text{str}_q(E_{SS}) = \text{sdim}_q(E_{SS}V^{\otimes t}) \neq 0$ and thus that $E_{SS} \notin \mathcal{J}_t$, proving the lemma.

We now show that any such projector E_{SS} belongs to \mathcal{J}_t . To see this, assume the contrary, then

$$E_{SS} = \widetilde{E}_{SS} + E_j, \quad (59)$$

where $\widetilde{E}_{SS} \in \Upsilon(\widetilde{BW}_t(-q^{2n}, q))$ and E_j is some (potentially vanishing) element of \mathcal{J}_t from (51). Explicitly,

$$\widetilde{E}_{SS} = \sum_{(R,T) \in \Omega_t(-q^{2n}, q)} c_{RT} \Upsilon(e_{RT}), \quad c_{RT} \in \mathbb{C}, \quad (60)$$

where the assumption that $E_{SS} \notin \mathcal{J}_t$ means that $c_{RT} \neq 0$ for at least one pair $(R, T) \in \Omega_t(-q^{2n}, q)$. Assume that $(A, B) \in \Omega_t(-q^{2n}, q)$ is a pair where $c_{AB} \neq 0$, then

$$\text{str}_q(\Upsilon(e_{BA})\widetilde{E}_{SS}) = \text{str}_q\left(\sum_T c_{AT} \Upsilon(e_{BT})\right) = c_{AB} \text{str}_q(\Upsilon(e_{BB})) \neq 0, \quad (61)$$

where the sum is over all paths T of length t in the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$, as $c_{AB} \neq 0$ and $\text{str}_q(\Upsilon(e_{BB})) \neq 0$.

However, (61) is not true. To see this, recall that E_{SS} satisfies

$$\Upsilon(e_{BA})E_{SS} = \Upsilon(e_{BA})(\widetilde{E}_{SS} + E_j) = 0, \quad (62)$$

and note that

$$\Upsilon(e_{BA})\widetilde{E}_{SS} \in \Upsilon(\widetilde{BW}_t(-q^{2n}, q)) \quad (63)$$

and that $(\Upsilon(e_{BA})E_j) \in \mathcal{J}_t$, which we recall arises from the definition of \mathcal{J}_t . Now from (63) and (51), $\Upsilon(e_{BA})\widetilde{E}_{SS} \neq -\Upsilon(e_{BA})E_j$ unless $\Upsilon(e_{BA})\widetilde{E}_{SS} = 0 = -\Upsilon(e_{BA})E_j$ and then (62) implies that $\Upsilon(e_{BA})\widetilde{E}_{SS} = \Upsilon(e_{BA})E_j = 0$.

It follows from the fact that $\Upsilon(e_{BA})\widetilde{E}_{SS} = 0$ that $\text{str}_q(\Upsilon(e_{BA})\widetilde{E}_{SS}) = 0$, however, this contradicts (61). Then the assumption in (60) that $c_{RT} \neq 0$ for at least one pair $(R, T) \in \Omega_t(-q^{2n}, q)$ is false, thus it must be true that $\widetilde{E}_{SS} = 0$.

It then follows that $E_{SS} = E_j \in \mathcal{J}_t$ from (59). However, this is false as $\text{str}_q(E_{SS}) = \text{sdim}_q(E_{SS}V^{\otimes t}) \neq 0$. Thus, our original assumption that there exists a projector $E_{SS} \in \mathcal{C}_t$ with the property that $(E_{SS}V^{\otimes t}) \cap (\Upsilon(e_{RR})V^{\otimes t}) = 0$ for all idempotent matrix units $e_{RR} \in \widetilde{BW}_t(-q^{2n}, q)$ is not true. This completes the proof. \square

11.5 Matrix units in \mathcal{C}_t

We have not yet proved that $\mathcal{L}_t = \mathcal{C}_t$. We will complete its proof in this subsection by defining a complete set of intertwiners in \mathcal{C}_t that we obtain by applying the map Υ to the intertwiner matrix units in $\widetilde{BW}_t(-q^{2n}, q)$.

There is a canonical bijection between paths in the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$ and paths in the Bratteli diagram for $V^{\otimes t}$. Recall that each Young diagram on the k^{th} level of the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$ contains an even (resp. odd) number of boxes if k is an even (resp. odd) number. Each vertex λ on the k^{th} level of the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$ also appears on the k^{th} level of the Bratteli diagram for $V^{\otimes t}$ unless λ has more than n rows of boxes, in which case the Young diagram $\tilde{\lambda}$, defined in Lemma 11.4, appears instead.

Given a path \tilde{S} of length t in the Bratteli diagram for $V^{\otimes t}$, we now determine the corresponding path S of length t in the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$. Let $\tilde{S} = (0, s_1, \dots, s_t)$ where s_i is a Young diagram on the i^{th} level of the Bratteli diagram for $V^{\otimes t}$. If i is even (resp. odd) and s_i contains an even (resp. odd) number of boxes, then s_i is also a vertex on the i^{th} level of the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$. If, however, i is even (resp. odd) and s_i contains an odd (resp. even) number of boxes, then $s_i = \tilde{\lambda}$ is the vertex obtained from a vertex λ on the i^{th} level of the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$ given by $\tilde{\lambda}'_1 = 2n + 1 - \lambda'_1$ and $\tilde{\lambda}'_j = \lambda'_j$ for $j \geq 2$. This allows us to define the path S of length t in the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$ corresponding to \tilde{S} and it also gives rise to the canonical bijection between the set of paths of length t in the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$ and the set of paths of length t in the Bratteli diagram for $V^{\otimes t}$.

In Figure 11.5 we show the isomorphism between the Bratteli diagram for $\widetilde{BW}_t(-q^2, q)$ up to the 4^{th} level (on the left) with the Bratteli diagram for $V^{\otimes t}$ up to the 4^{th} level (on the right), where V is the 3-dimensional irreducible $U_q(\text{osp}(1|2))$ -module.

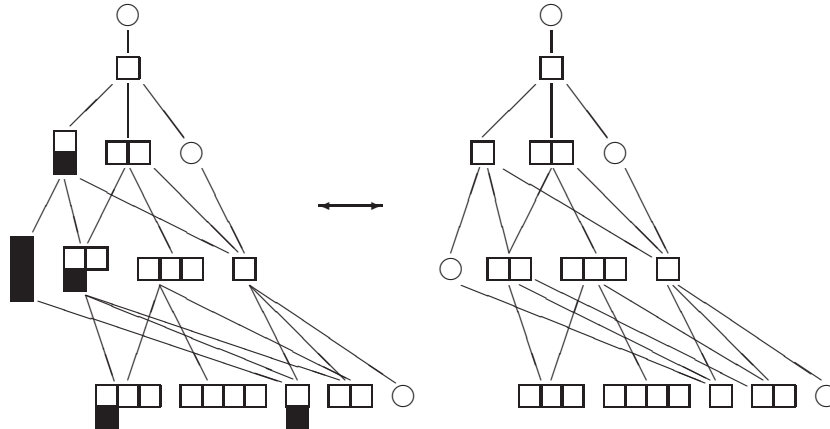


Figure 5: The isomorphism between the Bratteli diagram for $\widetilde{BW}_t(-q^2, q)$ up to the 4^{th} level (on the left) with the Bratteli diagram for $V^{\otimes t}$ up to the 4^{th} level (on the right), where V is the 3-dimensional irreducible $U_q(\text{osp}(1|2))$ -module

The diagram on the left in Figure 11.5 is the Bratteli diagram for $\widetilde{BW}_t(-q^2, q)$ up to the 4^{th}

level with some of the boxes of the vertices filled in. If λ is a vertex in the Bratteli diagram for $\widetilde{BW}_t(-q^2, q)$, then $\tilde{\lambda}$, as given in Lemma 11.4, is precisely the Young diagram λ with the filled in boxes removed.

We obtain the intertwiners in \mathcal{C}_t between the isomorphic irreducible $U_q(\mathfrak{g})$ -submodules of $V^{\otimes t}$ defined by the projectors $E_{RR} \in \mathcal{C}_t$ where R is a path of length t in the Bratteli diagram for $V^{\otimes t}$, by applying the homomorphism Υ to the intertwiner matrix units in $\widetilde{BW}_t(-q^{2n}, q)$. All we need to do is check that the images of the intertwiners under Υ are all well-defined and non-zero.

We construct the intertwiners in \mathcal{C}_t recursively. To do this, assume that all the matrix units in \mathcal{C}_{t-1} have already been defined and that they are non-zero. This condition is satisfied for $t = 3$ as the decomposition of $V \otimes V$ into irreducible $U_q(\mathfrak{g})$ -submodules is multiplicity free, thus no intertwiners exist in \mathcal{C}_2 .

In the remainder of this subsection, let \widetilde{M} and \widetilde{P} be a pair of paths of length t in the Bratteli diagram for $V^{\otimes t}$ where $\text{shp}(\widetilde{M}) = \text{shp}(\widetilde{P})$ and $\widetilde{M} \neq \widetilde{P}$. Let M and P be the corresponding paths in the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$. The intertwiner $E_{\widetilde{M}\widetilde{P}} \in \mathcal{C}_t$ is defined by $E_{\widetilde{M}\widetilde{P}} = \Upsilon(e_{MP})$.

Let us firstly deal with the situation that $\text{shp}(M) = \text{shp}(P) = \lambda$ where λ contains strictly fewer than t boxes. Referring back to Subsection 11.1 we see that

$$E_{\widetilde{M}\widetilde{P}} = \Upsilon(e_{MP}) = \frac{Q_\lambda(-q^{2n}, q)}{\sqrt{Q_\mu(-q^{2n}, q)Q_{\tilde{\mu}}(-q^{2n}, q)}} E_{\widetilde{M}'\widetilde{S}} \Upsilon(e_{t-1}) E_{\widetilde{T}\widetilde{P}'},$$

where $e_{t-1} \in BW_t(-q^{2n}, q)$,

where S and T are paths of length $t - 1$ in the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$ satisfying the conditions:

- (i) $\text{shp}(S) = \text{shp}(M') = \mu$, and
- (ii) $\text{shp}(T) = \text{shp}(P') = \tilde{\mu}$, and
- (iii) $S' = T'$, and
- (iv) $\text{shp}(S') = \lambda = \text{shp}(T')$.

Recall that these paths exist. Note that $Q_\mu(-q^{2n}, q) \neq 0$ and $Q_{\tilde{\mu}}(-q^{2n}, q) \neq 0$; if it were true that $Q_\mu(-q^{2n}, q) = 0$ then μ would not be a vertex in the Bratteli diagram for $\widetilde{BW}_t(-q^{2n}, q)$. Similar remarks hold for $\tilde{\mu}$ and λ . Thus $E_{\widetilde{M}\widetilde{P}}$ is well-defined. (We will later show that $E_{\widetilde{M}\widetilde{P}}$ is non-zero.)

Now let us deal with the situation that $\text{shp}(M) = \text{shp}(P) = \lambda$ where λ contains exactly t boxes and $\text{shp}(M') \neq \text{shp}(P')$. Referring back to Subsection 11.1, we see that

$$E_{\widetilde{M}\widetilde{P}} = \Upsilon(e_{MP}) = \Upsilon((1 - z_t)o_{MP}),$$

where $o_{MP} = o_{M'P'}o_{PP}$ and $z_t = \sum_S e_{SS}$ with the summation going over all paths S of length t such that $\text{shp}(S)$ contains fewer than t boxes.

Now let us deal with the situation that $\text{shp}(M) = \text{shp}(P) = \lambda$ where λ contains exactly t boxes and $\text{shp}(M') \neq \text{shp}(P')$. Choose paths \overline{M} and \overline{P} of length t such that $\text{shp}(\overline{M}) = \text{shp}(M)$ and $\text{shp}(\overline{P}) = \text{shp}(P)$ and

- (i) $\overline{M}'' = \overline{P}''$, and
- (ii) $\text{shp}(\overline{M}') = \text{shp}(M')$, and
- (iii) $\text{shp}(\overline{P}') = \text{shp}(P')$.

Such paths can always be chosen. Then

$$E_{\widetilde{M}\widetilde{P}} = \Upsilon(e_{MP}) = \Upsilon((1 - z_t)o_{MP}), \quad \text{where}$$

$$o_{MP} = \frac{1 - q^{2d}}{\sqrt{(1 - q^{2(d+1)})(1 - q^{2(d-1)})}} o_{M'\overline{M}'} g_{t-1} o_{\overline{P}'} o_{PP}, \quad g_{t-1} \in BW_t(-q^{2n}, q), \quad (64)$$

where $d = d(\overline{M}, t-1)$ is the integer defined by (45). The integer $|d(\overline{M}, i)| + 1$ is the number of boxes in the hook going through the boxes containing the numbers i and $(i+1)$ [22].

We now prove that the coefficient on the right hand side of (64) is well-defined and non-zero. It is not difficult to see that the coefficient is well-defined if $|d| \neq 1$, and we now show that this is always true. As $|d| + 1$ is the length of the hook going through the boxes containing the numbers $(t-1)$ and t , it is always true that $|d| + 1 \geq 2$ as each such hook contains at least two boxes. Now the only situation in which it could possibly be true that $|d| = 1$ is if the boxes containing the numbers $(t-1)$ and t are immediately horizontally or vertically adjacent. However this cannot occur for the following reason: from the above construction, the number t is in the same box in \overline{M} as the number $(t-1)$ is in \overline{P} , and the number t is in the same box in \overline{P} that the number $(t-1)$ is in \overline{M} . It follows that if the numbers $(t-1)$ and t are immediately horizontally or vertically adjacent in \overline{M} , each must be in the corresponding ‘swapped’ box in \overline{P} , and then at least one of \overline{M} or \overline{P} cannot be a standard tableau. This contradicts the assumption that both \overline{M} and \overline{P} are standard tableaux, thus $|d| \neq 1$ and the coefficient in (64) is well-defined.

It remains for us to show that the coefficient in (64) is non-zero. This follows immediately from the fact that $|d| \neq 0$. Note that we have not yet proved that the matrix units are all non-zero.

Let us write E_{MP} to denote $E_{\overline{M}\overline{P}}$. We note that the matrix unit $E_{MP} \in \mathcal{C}_t$, for $M \neq P$, is an intertwiner between the isomorphic irreducible $U_q(\mathfrak{g})$ -modules $E_{PP}(V^{\otimes t})$ and $E_{MM}(V^{\otimes t})$:

$$E_{MP} : E_{PP}(V^{\otimes t}) \rightarrow E_{MM}(V^{\otimes t}),$$

and that the whole collection of matrix units satisfy

$$E_{QR}E_{ST} = \delta_{RS}E_{QT}.$$

To show that each intertwiner E_{MP} is non-zero, it suffices to note that each projector E_{PP} is non-zero and that $E_{PP} = E_{PM}E_{MP}$.

We then have the complete set of projectors and intertwiners in \mathcal{C}_t . This means that $\mathcal{L}_t = \mathcal{C}_t$. To see this, note that the matrix units $\{E_{ST} = \Upsilon(e_{ST}) \mid (S, T) \in \Omega_t(-q^{2n}, q)\}$ are a basis for \mathcal{C}_t , as the fact that the superdimension of each finite dimensional irreducible $U_q(\mathfrak{g})$ -module V_λ with integral dominant highest weight λ is non-zero means that Schur’s lemma has the same form as it does for ungraded quantum algebras, thus the centraliser \mathcal{L}_t is generated by the complete set $\{E_{ST} = \Upsilon(e_{ST}) \mid (S, T) \in \Omega_t(-q^{2n}, q)\}$ of projectors and intertwiners.

Note that $\mathcal{J}_t = 0$; to see this, let X be an arbitrary element of \mathcal{C}_t , then

$$X = \sum_{(S,T) \in \Omega_t(-q^{2n}, q)} x_{ST} E_{ST}, \quad x_{ST} \in \mathbb{C},$$

where $x_{ST} \neq 0$ for at least one pair (S, T) . Let (A, B) be such a pair, then

$$\text{str}_q(E_{BA}X) = \text{str}_q(x_{AB}E_{BA}E_{AB}) = x_{AB}\text{str}_q(E_{BB}) \neq 0,$$

thus $X \notin \mathcal{J}_t$. As X was arbitrary, $\mathcal{J}_t = 0$.

Note that we obtained $\mathcal{C}_t \equiv \mathcal{L}_t$ by using the fact that λ and $\tilde{\lambda}$ do not appear on the same level of the Bratteli diagram for $BW_t(-q^{2n}, q)$. If λ and $\tilde{\lambda}$ did appear on the same level, we could only conclude that there is an inclusion of \mathcal{C}_t in \mathcal{L}_t rather than an equality. Of course, in that event, there may actually be an equality, but a different method would have to be used to obtain all the intertwiners.

We now present the technical lemmas used above.

Lemma 11.7. *Let $\psi : \mathcal{C}_t \rightarrow \mathbb{C}$ be a map defined by*

$$\psi(X) = \frac{\text{str}_q(X)}{(\text{sdim}_q(V))^t},$$

and let tr be the trace functional on $BW_t(-q^{2n}, q)$ mentioned in (47). Then

$$\psi(\Upsilon(a)) = \text{tr}(a), \quad \forall a \in BW_t(-q^{2n}, q).$$

Proof. Any functional ϕ on $BW_\infty(-q^{2n}, q)$ satisfying Eq. (47) for all $t \in \mathbb{N}$ is identical to tr [24, Lem. 3.4 (d)], and we will show that $\psi \circ \Upsilon$ is such a functional.

To show that $\psi \circ \Upsilon$ satisfies Eq. (47), it suffices to show that for each $t \in \mathbb{N}$,

$$-\psi(\Upsilon(a)\check{R}_{t-1}\Upsilon(b)) = -\psi(\check{R}_{t-1})\psi(\Upsilon(ab)), \quad \forall a, b \in BW_{t-1}(-q^{2n}, q), \quad (65)$$

as the element $e_{t-1} \in BW_t(-q^{2n}, q)$ can be written as a function of the g_{t-1} 's. We will show that Eq. (65) is true using Lemmas 11.8 and 11.9.

The left hand side of Eq. (65) is

$$-\text{str}_q^{\otimes t}(A\check{R}_{t-1}B)/(sdim_q(V))^t, \quad (66)$$

where $\text{str}_q^{\otimes t}$ indicates that we take the quantum supertrace over all t tensor factors, and we also write $A = \Upsilon(a)$ and $B = \Upsilon(b)$. Now we can regard each $X \in \mathcal{C}_{t-1}$ as an element of \mathcal{C}_t under the mapping $X \mapsto X \otimes \text{id}$, then by applying the identity to the first $t-1$ tensor powers of (66) and taking the quantum supertrace over the t^{th} tensor power of (66), we obtain, using Lemmas 11.8 and 11.9,

$$-\text{str}_q^{\otimes t}(A\check{R}_{t-1}B)/(sdim_q(V))^t = \frac{-\chi_V(v^{-1})}{sdim_q(V)} \frac{\text{str}_q^{\otimes(t-1)}(AB)}{(sdim_q(V))^{t-1}}. \quad (67)$$

Now

$$\psi(\check{R}_{t-1}) = \chi_V(v^{-1})/sdim_q(V),$$

and the right hand side of (67) equals the right hand side of (65). Now (65) is true for all a and b belonging to $BW_{t-1}(-q^{2n}, q)$, and it remains to show that $\psi \circ \Upsilon$ is a functional on $BW_\infty(-q^{2n}, q)$ satisfying (47) for all $t = 1, 2, \dots$. This follows from the fact that $\psi(A \otimes \text{id}) = \psi(A)$ for all $A \in \mathcal{C}_t$, thus we can regard ψ as well-defined in the inductive limit $\mathcal{C}_2 \subset \mathcal{C}_3 \subset \mathcal{C}_4 \subset \dots$, which completes the proof. \square

The following lemma, which we used in the proof of Lemma 11.7, appears in [18, Lem. 2] and is proved in [26, Lem. 3.1].

Lemma 11.8. *Let V be the fundamental $(2n+1)$ -dimensional irreducible $U_q(\mathfrak{osp}(1|2n))$ -module with highest weight ϵ_1 and let π be the representation of $U_q(\mathfrak{osp}(1|2n))$ afforded by V . Let $\check{\mathcal{R}}_{V,V} \in \text{End}_{U_q(\mathfrak{osp}(1|2n))}(V \otimes V)$ be as given in Eq. (33). Then*

$$(\text{id} \otimes \text{str}) \left[(\text{id} \otimes \pi(K_{2\rho})) \check{\mathcal{R}}_{V,V}^{\pm 1} \right] = q^{\pm(\epsilon_1, \epsilon_1 + 2\rho)} \text{id}_V = \chi_V(v^{\mp 1}) \text{id}_V.$$

Lemma 11.9. *For all $x, y \in \mathcal{C}_t$,*

$$(\text{id}^{\otimes t} \otimes \text{str}_q) x \check{R}_t y = [(\text{id}^{\otimes t} \otimes \text{str}_q) \check{R}_t] xy, \quad (68)$$

where each element $z \in \mathcal{C}_t$ is embedded in \mathcal{C}_{t+1} via the mapping $z \mapsto z \otimes \text{id}$.

Proof. Let W be a finite dimensional integrable $U_q(\mathfrak{osp}(1|2n))$ -module and V be the $(2n+1)$ -dimensional irreducible $U_q(\mathfrak{osp}(1|2n))$ -module. Let $\{w^i \mid i \in I\}$ be a homogeneous basis of W and $\{v^j \mid j \in J\}$ be the homogeneous basis of V used throughout this paper where $v^j = v_j$.

We write the action of $B \in \text{End}_{\mathbb{C}}(W)$ as $Bw^i = \sum_{k \in I} B_i^k w^k$, $B_i^k \in \mathbb{C}$, and the action of $A \in \text{End}_{\mathbb{C}}(W \otimes V)$ as

$$A(w^i \otimes v^j) = \sum_{k \in I, l \in J} A_{ij}^{kl} w^k \otimes v^l, \quad A_{ij}^{kl} \in \mathbb{C},$$

with the obvious generalisations to further tensor powers. For such an A , $[(\text{id} \otimes \text{tr})A] \in \text{End}_{\mathbb{C}}(W)$ with the action

$$[(\text{id} \otimes \text{tr})A]_k^i = \sum_{j \in J} A_{kj}^{ij}.$$

Similarly $[(\text{id} \otimes \text{str})A]_k^i = \sum_{j \in J} (-1)^{[v^j]} A_{kj}^{ij}$ and

$$[(\text{id} \otimes \text{str}_q)A]_k^i = \sum_{j \in J} (-1)^{[v^j]} \langle v^j, K_{2\rho} v^j \rangle A_{kj}^{ij},$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing: $\langle v^j, v^k \rangle = \delta_{jk}$, and $[v^j] = [v^j]$.

Clearly,

$$(\text{id}^{\otimes t} \otimes \text{str}_q) \check{R}_t = \chi_V(v^{-1})(\text{id}_V)^{\otimes t}, \quad (69)$$

from Lemma 11.8. Now let

$$\left\{ v^{\vec{i}} = v^{i_1} \otimes v^{i_2} \otimes \cdots \otimes v^{i_{t-1}} \mid \vec{i} = (i_1, i_2, \dots, i_{t-1}) \in \bar{J} = J^{\times(t-1)} \right\}$$

be a homogeneous basis of $V^{\otimes(t-1)}$, then we can write $x \in \mathcal{C}_t$ as

$$x = \left(X_{\vec{j}l}^{\vec{i}k} \right)_{\vec{i}, \vec{j} \in \bar{J}, k, l \in J} \quad \text{and} \quad x \otimes \text{id}_V = \left(X_{\vec{j}ls}^{\vec{i}kr} \right)_{\vec{i}, \vec{j} \in \bar{J}, k, l, r, s \in J}$$

where

$$X_{\vec{j}ls}^{\vec{i}kr} = \delta_{rs} X_{\vec{j}l}^{\vec{i}k}. \quad (70)$$

In this notation, we can rewrite (69): let

$$\check{R}_t = \left(R_{\vec{j}ls}^{\vec{i}kr} \right)_{\vec{i}, \vec{j} \in \bar{J}, k, l, r, s \in J}$$

then we have

$$[(\text{id}^{\otimes t} \otimes \text{str}_q) \check{R}_t]_{\vec{j}l}^{\vec{i}k} = \sum_p (-1)^{[v^p]} \langle v^p, K_{2\rho} v^p \rangle R_{\vec{j}lp}^{\vec{i}kp} = \delta_{\vec{i}\vec{j}} \delta_{kl} \chi_V(v^{-1}). \quad (71)$$

Now let x and y be arbitrary elements of \mathcal{C}_t embedded in \mathcal{C}_{t+1} via the map $z \mapsto z \otimes \text{id}_V$. To determine the left hand side of (68), we note that

$$\begin{aligned} [(x \otimes \text{id}_V) \check{R}_t (y \otimes \text{id}_V)]_{\vec{j}ls}^{\vec{i}kr} &= \sum_{\vec{a}, \vec{d} \in \bar{J}, b, c, e, f \in J} X_{\vec{a}bc}^{\vec{i}kr} R_{\vec{d}ef}^{\vec{a}bc} Y_{\vec{j}ls}^{\vec{d}ef} \\ &= \sum_{\vec{a}, \vec{d} \in \bar{J}, b, e \in J} X_{\vec{a}br}^{\vec{i}kr} R_{\vec{d}es}^{\vec{a}br} Y_{\vec{j}ls}^{\vec{d}es}, \end{aligned}$$

from (70). For arbitrary $\vec{i}, \vec{j} \in \bar{J}$ and $k, l \in J$, we have

$$\begin{aligned} [(\text{id}^{\otimes t} \otimes \text{str}_q) x \check{R}_t y]_{\vec{j}l}^{\vec{i}k} &= \sum_{p \in J} (-1)^{[v^p]} \langle v^p, K_{2\rho} v^p \rangle [(x \otimes \text{id}_V) \check{R}_t (y \otimes \text{id}_V)]_{\vec{j}lp}^{\vec{i}kp} \\ &= \sum_{p \in J} (-1)^{[v^p]} \langle v^p, K_{2\rho} v^p \rangle \sum_{\vec{a}, \vec{d} \in \bar{J}, b, c \in J} X_{\vec{a}bp}^{\vec{i}kp} R_{\vec{d}cp}^{\vec{a}bp} Y_{\vec{j}lp}^{\vec{d}cp} \\ &= \sum_{\vec{a}, \vec{d} \in \bar{J}, b, c \in J} X_{\vec{a}b}^{\vec{i}k} \left(\sum_p (-1)^{[v^p]} \langle v^p, K_{2\rho} v^p \rangle R_{\vec{d}cp}^{\vec{a}bp} \right) Y_{\vec{j}l}^{\vec{d}c}. \end{aligned} \quad (72)$$

The sum over p inside the brackets in (72) equals $\delta_{\vec{a}\vec{d}} \delta_{bc} \chi_V(v^{-1})$ from (71), thus (72) simplifies to

$$\chi_V(v^{-1}) \sum_{\vec{a} \in \bar{J}, b \in J} X_{\vec{a}b}^{\vec{i}k} Y_{\vec{j}l}^{\vec{a}b} = \chi_V(v^{-1}) [xy]_{\vec{j}l}^{\vec{i}k},$$

proving the lemma. \square

12 A representation of the Hecke algebra $H_t(q)$ of type A_{t-1}

The structures of the centraliser algebras of tensor products of finite dimensional irreducible spinorial representations of $U_q(\mathfrak{g})$ is not known. Musson and Zou presented branching rules for tensoring a finite dimensional irreducible $U_q(\mathfrak{g})$ -module with a spinorial representation in [20, sec. 5], but a complete understanding is lacking.

In this section, we make an observation that has not, to the author's knowledge, appeared in the literature. Let $V_{1/2}^+$ be the two-dimensional irreducible representation of $U_q(\mathfrak{osp}(1|2))$ with even highest weight vector, then there exists a representation of $H_t(-q)$ in $\text{End}_{U_q(\mathfrak{osp}(1|2))} \left[(V_{1/2}^+)^{\otimes t} \right]$.

12.1 $U_q(\mathfrak{osp}(1|2))$ and its finite-dimensional irreducible representations

We fix $\mathfrak{g} = \mathfrak{osp}(1|2)$ in this section. Recall the definition of $U_q(\mathfrak{g})$. The generators e and f are odd and $K^{\pm 1}$ are even.

Let v_+ be a non-zero vector satisfying

$$ev_+ = 0, \quad Kv_+ = \omega v_+, \quad \omega \in \mathbb{C}.$$

We assume in the rest of this section that q is generic. It is not difficult to prove that the highest weight vector v_+ generates a finite-dimensional irreducible highest weight $U_q(\mathfrak{g})$ -module $V(\omega)$ if and only if

$$\omega = \begin{cases} \pm q^\lambda, & \lambda \in \mathbb{Z}_+, \\ \pm iq^\lambda, & \lambda \in \mathbb{Z}_+ + 1/2. \end{cases}$$

In both of these cases, $V(\omega)$ is $(2\lambda + 1)$ -dimensional and we label $V(\omega)$ by V_λ . To be more precise, we write V_λ^+ (resp. V_λ^-) where $Kv_+ = q^\lambda$ or $Kv_+ = iq^\lambda$ (resp. $Kv_+ = -q^\lambda$ or $Kv_+ = -iq^\lambda$).

In contrast with the rest of this paper, we take the grading of the highest weight vectors of V_λ^\pm to be even in this section, and denote by $\Pi V_\lambda^{\pm 1}$ the corresponding irreducible $(2\lambda + 1)$ -dimensional $U_q(\mathfrak{g})$ -modules with odd highest weight vectors.

The tensorial (odd-dim.) irreducible representations of $U_q(\mathfrak{g})$ are deformations of irreducible representations of $U(\mathfrak{g})$. The spinorial representations are not deformations of representations of $U(\mathfrak{g})$ as $U(\mathfrak{g})$ admits no irreducible even-dimensional representations.

It is easy to prove the following branching rules:

Lemma 12.1.

$$V_\lambda^+ \otimes V_{1/2}^+ \cong \begin{cases} V_{\lambda+1/2}^+ \oplus \Pi V_{\lambda-1/2}^+, & \text{if } \lambda \in \{1, 2, \dots\}, \\ V_{\lambda+1/2}^- \oplus \Pi V_{\lambda-1/2}^-, & \text{if } \lambda \in \{1/2, 3/2, 5/2, \dots\}. \end{cases} \quad (73)$$

We often employ the two-dimensional irreducible $U_q(\mathfrak{g})$ -module $V_{1/2}^+$ in this section. From (73) it is convenient to define a Bratteli diagram for $(V_{1/2}^+)^{\otimes t}$ similar to the Bratteli diagram for tensor products of the two-dimensional irreducible representation of $U_q(\mathfrak{sl}(2))$: let the t^{th} level of the Bratteli diagram for $(V_{1/2}^+)^{\otimes t}$ consist of all Young diagrams μ with t boxes arranged in at most two rows: $\mu = [\mu_1, \mu_2]$, then μ labels the highest weight of an irreducible $U_q(\mathfrak{g})$ -module $V_{(\mu_1 - \mu_2)/2}^\pm$. A vertex μ on the m^{th} row is connected to a vertex ν on the $(m + 1)^{\text{st}}$ row if and only if μ and ν differ by exactly one box.

It is interesting to observe from (73) that for each fixed t , all the irreducible $U_q(\mathfrak{g})$ -summands of $(V_{1/2}^+)^{\otimes t}$ are labelled with the same superscript ('+' or '-'); and that the superscript is '+' if $t \equiv 0, 1 \pmod{4}$ and '-' otherwise. Furthermore, if V_λ^\pm (resp. ΠV_λ^\pm) is a summand in $(V_{1/2}^+)^{\otimes t}$, then ΠV_λ^\pm (resp. V_λ^\pm) is not a summand in $(V_{1/2}^+)^{\otimes t}$.

12.2 R -matrix for $V_{1/2}^+ \otimes V_{1/2}^+$

We now write down the R -matrix for $\text{End}_{\mathbb{C}}(V_{1/2}^+ \otimes V_{1/2}^+)$ and thereby obtain a representation of the Braid group B_t on t strings in $\text{End}_{U_q(\mathfrak{g})} \left[(V_{1/2}^+)^{\otimes t} \right]$, which also gives a representation of the Hecke algebra $H_t(-q)$ of type A_{t-1} .

To write down an R -matrix for $\text{End}_{\mathbb{C}}(V_{1/2}^+ \otimes V_{1/2}^+)$ we need to slightly change the method we used to write down R -matrices for representations of $U_q(\text{osp}(1|2n))$ previously. In particular, we take the highest weight of $V_{1/2}^+$ to be complex: write $q = e^{i\phi}$, $\phi \in \mathbb{C}$ and let v_+ be the highest weight vector of $V_{1/2}^+$, then

$$Kv_+ = iq^{1/2}v_+ = q^{1/2+\pi/2\phi}v_+.$$

Let $\{v_{1/2}, v_{-1/2}\}$ be a homogeneous basis of $V_{1/2}^+$ where $v_{1/2} = v_+$ and $v_{-1/2} = fv_+$. Noting that $V_{1/2}^+$ is two-dimensional, let

$$\tilde{R}^V = 1 \otimes 1 + (q^{-1} - q)(e \otimes f)$$

and define $\mathcal{E} \in \text{End}_{\mathbb{C}}(V_{1/2}^+ \otimes V_{1/2}^+)$ by

$$\mathcal{E}(v_j \otimes v_k) = q^{(j+\pi/2\phi)(k+\pi/2\phi)}(v_j \otimes v_k), \quad j, k \in \{1/2, -1/2\},$$

then upon writing π to denote the representation of $U_q(\mathfrak{g})$ afforded by $V_{1/2}^+$ and writing $\mathcal{R}_{\frac{1}{2}\frac{1}{2}} = \mathcal{E} \cdot (\pi \otimes \pi) \tilde{R}^V$, it can be shown, in a similar way as we did previously for tensorial representations, that

$$\mathcal{R}_{\frac{1}{2}\frac{1}{2}} \cdot [(\pi \otimes \pi)\Delta(x)] = [(\pi \otimes \pi)\Delta'(x)] \cdot \mathcal{R}_{\frac{1}{2}\frac{1}{2}}, \quad \forall x \in U_q(\mathfrak{g}).$$

A similar process can be followed to create R -matrices for tensor products of other irreducible spinorial representations of $U_q(\mathfrak{g})$.

A standard argument shows that the even map $\tilde{\mathcal{R}}_{\frac{1}{2}\frac{1}{2}} \in \text{End}_{\mathbb{C}}(V_{1/2}^+ \otimes V_{1/2}^+)$ defined by

$$\tilde{\mathcal{R}}_{\frac{1}{2}\frac{1}{2}}(v_j \otimes v_k) = P \circ \left(\mathcal{R}_{\frac{1}{2}\frac{1}{2}}(v_j \otimes v_k) \right), \quad j, k \in \{1/2, -1/2\},$$

where P is the graded permutation operator, is $U_q(\mathfrak{g})$ -linear:

$$\tilde{\mathcal{R}}_{\frac{1}{2}\frac{1}{2}} \cdot [(\pi \otimes \pi)\Delta(x)] = [(\pi \otimes \pi)\Delta(x)] \cdot \tilde{\mathcal{R}}_{\frac{1}{2}\frac{1}{2}}, \quad \forall x \in U_q(\mathfrak{g}).$$

With respect to the ordered basis $\{v_{1/2} \otimes v_{1/2}, v_{1/2} \otimes v_{-1/2}, v_{-1/2} \otimes v_{1/2}, v_{-1/2} \otimes v_{-1/2}\}$, $\tilde{\mathcal{R}}_{\frac{1}{2}\frac{1}{2}}$ is explicitly

$$\tilde{\mathcal{R}}_{\frac{1}{2}\frac{1}{2}} = i^{\pi/2\phi} \begin{pmatrix} iq^{1/4} & 0 & 0 & 0 \\ 0 & 0 & q^{-1/4} & 0 \\ 0 & q^{-1/4} & i(q^{1/4} + q^{-3/4}) & 0 \\ 0 & 0 & 0 & iq^{1/4} \end{pmatrix}.$$

Recall from (73) that $V_{1/2}^+ \otimes V_{1/2}^+ \cong V_1^- \oplus \Pi V_0^-$, then a direct calculation gives

$$\tilde{\mathcal{R}}_{\frac{1}{2}\frac{1}{2}} \cdot w_1 = i^{1+\pi/2\phi} q^{1/4} w_1, \quad \forall w_1 \in V_1^- \subset V_{1/2}^+ \otimes V_{1/2}^+,$$

$$\tilde{\mathcal{R}}_{\frac{1}{2}\frac{1}{2}} \cdot w_0 = i^{1+\pi/2\phi} q^{-3/4} w_0, \quad \forall w_0 \in \Pi V_0^- \subset V_{1/2}^+ \otimes V_{1/2}^+,$$

thus

$$\left(\tilde{\mathcal{R}}_{\frac{1}{2}\frac{1}{2}} - i^{1+\pi/2\phi} q^{1/4} \right) \left(\tilde{\mathcal{R}}_{\frac{1}{2}\frac{1}{2}} - i^{1+\pi/2\phi} q^{-3/4} \right) = 0.$$

By inspection,

$$(\text{id} \otimes \text{str}) \left[(\text{id} \otimes \pi(K)) \tilde{\mathcal{R}}_{\frac{1}{2}\frac{1}{2}}^{\pm 1} \right] = \chi_{V_{1/2}^+}(v^{\mp 1}) \pi(\text{id}), \quad (74)$$

where $\chi_{V_{1/2}^+}(v^{-1}) = -i^{\pi/2\phi} q^{3/4}$ and $\chi_{V_{1/2}^+}(v) = -i^{-\pi/2\phi} q^{-3/4}$.

Defining

$$\check{R}_j^{\pm 1} = \text{id}^{\otimes(j-1)} \otimes \check{R}_{\frac{1}{2}}^{\pm 1} \otimes \text{id}^{\otimes(t-j-1)} \in \text{End}_{U_q(\mathfrak{g})} \left[(V_{1/2}^+)^{\otimes t} \right],$$

and fixing $\mathcal{C}_t^{\frac{1}{2}}$ to be the complex algebra generated by $\{\check{R}_j^{\pm 1} \mid j = 1, 2, \dots, t-1\}$, we obtain a representation of the Braid group B_t via the homomorphism $\bar{\rho} : B_t \rightarrow \mathcal{C}_t^{\frac{1}{2}}$ defined by $\bar{\rho} : \sigma_j^{\pm 1} \mapsto \check{R}_j^{\pm 1}$. The map

$$\psi(X) = \frac{\text{str}_q(X)}{(\text{sdim}_q(V_{1/2}^+))^t}, \quad X \in \mathcal{C}_t^{\frac{1}{2}}$$

is a Markov trace and it should be possible to construct link invariants following [8] using this Markov trace, Eq. (74) and Lemma 11.9 adapted to this section.

12.3 A representation of the Hecke algebra $H_t(q)$

Following [23] we define the Hecke algebra $H_t(q)$ of type A_{t-1} to be the complex associative unital algebra generated by the elements $\{g_1, g_2, \dots, g_{t-1}\}$ subject to the relations

$$\begin{aligned} g_j g_{j+1} g_j &= g_{j+1} g_j g_{j+1}, & j &= 1, 2, \dots, t-1, \\ g_j g_k &= g_k g_j, & |j-k| &> 1, \\ (g_j)^2 &= (q-1)g_j + q, & j &= 1, 2, \dots, t-1, \end{aligned}$$

where q is a complex number. Then, we immediately have the following result:

Lemma 12.2. *The algebra homomorphism $\rho : H_t(-q) \rightarrow \mathcal{C}_t^{\frac{1}{2}}$ defined by*

$$\rho : g_j \mapsto i^{1-\pi/2\phi} q^{3/4} \check{R}_j,$$

yields a representation of $H_t(-q)$.

It may be possible to construct projectors and intertwiners in $\text{End}_{U_q(\mathfrak{g})} \left[(V_{1/2}^+)^{\otimes t} \right]$ by applying ρ to matrix units (eg those given in [22]) in $H_t(-q)$, but we have not gone through the details. As with the representations of the Birman-Wenzl-Murakami algebra we defined earlier in this paper, we expect that the representation of $H_t(-q)$ given in Lemma 12.2 is not faithful.

Furthermore, even if all the projectors onto and intertwiners between the irreducible $U_q(\mathfrak{g})$ -summands of $(V_{1/2}^+)^{\otimes t}$ can be obtained by applying ρ to matrix units in $H_t(-q)$, the fact that $V_{1/2}^+$ has vanishing superdimension may result in $\mathcal{C}_t^{\frac{1}{2}}$ being a proper subalgebra of $\text{End}_{U_q(\mathfrak{g})} \left[(V_{1/2}^+)^{\otimes t} \right]$ from Schur's lemma. However, we have not explored this issue and leave it for further study.

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